Assignment IV

Honours Linear Algebra TATA53 Spring 2024 Jonathan Nilsson

Instructions: Write your name, personal number, and program (if applicable) on the first page. The assignment should be solved individually. Your solutions should not depend on digital tools, except for problems marked by ⌨. Write complete solutions with clear answers. You are allowed to discuss the problems with other students, but you should write your solutions individually, don't write anything that you don't understand yourself. Write your solutions on paper by hand and staple them together. Some problems marked ⚔ are harder, it is not necessary to solve all the problems. You should be prepared to present your solutions on *one* of the two associated seminars.

Deadline: 10.00AM on May 13 2024. Hand in the assignments in the compartment labelled "TATA53 assignment submissions" one floor up in the B-building just above entrance 21, or bring them to the *first* seminar starting 15 minutes later.

1. Find the compact singular value decomposition $A = \tilde{U}\tilde{\Sigma}\tilde{V}^*$ of the matrix A below. Also find the matrix of rank 1 which is closest to A with respect to the Frobenius norm. [7]

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

2. Find the line in $\mathbb{R}^2$ that lies closest to the following 6 points (x, y) in the *total least squares* sense. [6]

x	5	1	3	6	4	5
y	4	3	2	3	1	5

3. For a 4×4-matrix A we know the Frobenius norm, the condition number, the nuclear norm, and the operator norm of the inverse of A: [6]

$$\|A\|_F = 10, \quad \kappa(A) = 4\sqrt{2}, \quad \|A\|_{\text{☢}} = 16 + \sqrt{2}, \quad \left\|A^{-1}\right\|_{\text{op}} = \frac{1}{\sqrt{2}}.$$

Find the matrix Σ in the singular value decomposition $A = U\Sigma V^*$ of A.

4. Consider the space $V = \mathrm{Mat}_2(\mathbb{R})$ with basis [6]

$$(e_1, e_2, e_3, e_4) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Let $f \in V^*$ be the trace function defined by $f(A) = \mathrm{tr}(A)$. Express f in the dual basis $(e_1^*, e_2^*, e_3^*, e_4^*)$ of V^*.

5. Find the polar factorization $A = UP$ of the matrix $A = \begin{pmatrix} 2 & 1 \\ 2 & -4 \end{pmatrix}$ where U is unitary and P is positive definite. Also determine the condition number $\kappa(A)$. [6]

6. Determine whether each statement below is true or false. Motivate with a proof or a counter-example. All matrices A and B below are of size $n \times n$. [6]

(a) If σ is a singular value of A, then $\sigma + 1$ is a singular value of $A + I$.

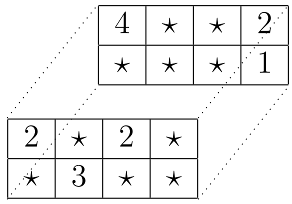
(b) Defining $\|A\| := \sigma_2(A)$ gives a norm on $\mathrm{Mat}_n(\mathbb{C})$.

(c) Given a QR-factorization $A = QR$, we always have $\kappa(A) = \kappa(R)$.

(d) The following vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^2 \otimes \mathbb{R}^2$ are linearly dependent:
$v_1 = \binom{1}{1} \otimes \binom{0}{1}, \quad v_2 = \binom{1}{0} \otimes \binom{1}{1}, \quad v_3 = \binom{1}{1} \otimes \binom{1}{0}, \quad v_4 = \binom{0}{1} \otimes \binom{1}{1}.$

(e) The Kronecker product satisfies $\det(A \otimes B) = (\det(A)\det(B))^n$

(f) The following 3d-array can be completed so that it corresponds to a *pure* tensor in $\mathbb{R}^2 \otimes \mathbb{R}^4 \otimes \mathbb{R}^2$:



7. The table below contains examination results from a random sample of 10 students from the Y-program at LiU. [6]

	1	2	3	4	5	6	7	8	9	10
Envariabelanalys I	11	8	9	13	9	12	10	12	12	11
Linjär algebra	10	12	12	15	11	15	14	14	14	10
Optimeringslära	11	8	8	18	12	9	19	18	15	15
Flervariabelanalys	10	9	14	14	12	13	13	19	14	8

Let $A \in \mathrm{Mat}_{10\times 4}(\mathbb{R})$ be the transpose of the matrix above. Perform principal component analysis on A by normalizing the data and computing the SVD, and then project the data onto the first two principal components. Produce two 2d-diagrams, one that shows the 10 students when projected onto the first two right singular vectors, and one that shows the four courses when projected onto the first two left singular vectors. Attach your code and its outputs. How do you interpret the first principal component?

8. (a) Find the Jordan form of the Kronecker product of Jordan blocks $J_2(1) \otimes J_3(1)$. [7]

(b) Derive the Clebsch-Gordan decomposition formula for linear maps: find a general formula for the Jordan form of

$$J_m(\lambda) \otimes J_n(\mu),$$

where λ and μ are nonzero. *Hint: start with the case $\lambda = \mu = 1$ and $m = 2$.*