TATA53 Lecture 2

Linear Algebra Honours course

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Part I

Direct sum

Direct sum (internal)

Let U_1 and U_2 be subspaces of V. If every $v \in V$ can be expressed uniquely as

$$v = u_1 + u_2 \text{ where } u_1 \in U_1,\ u_2 \in U_2,$$

then we write $V = U_1 \oplus U_2$.

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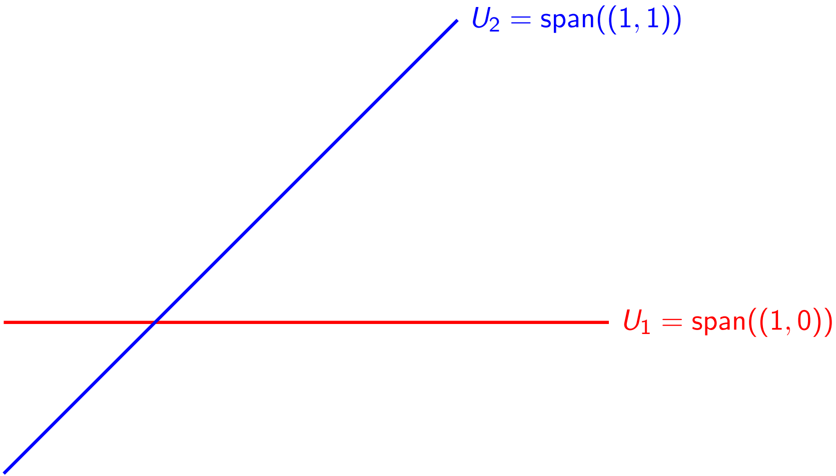
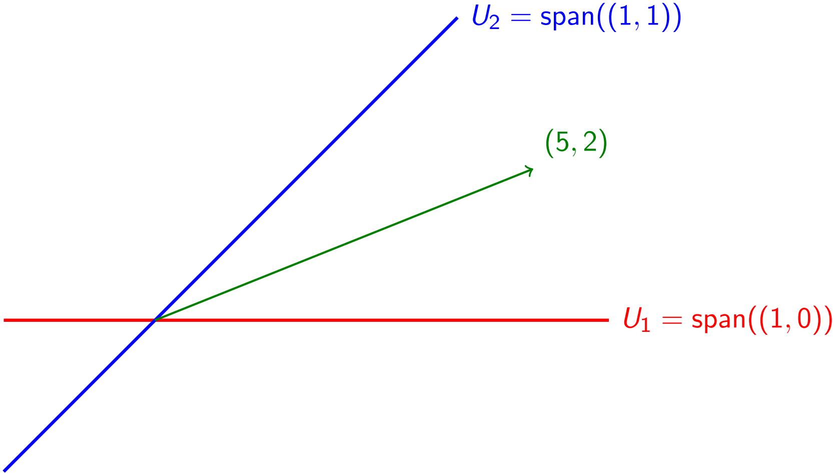
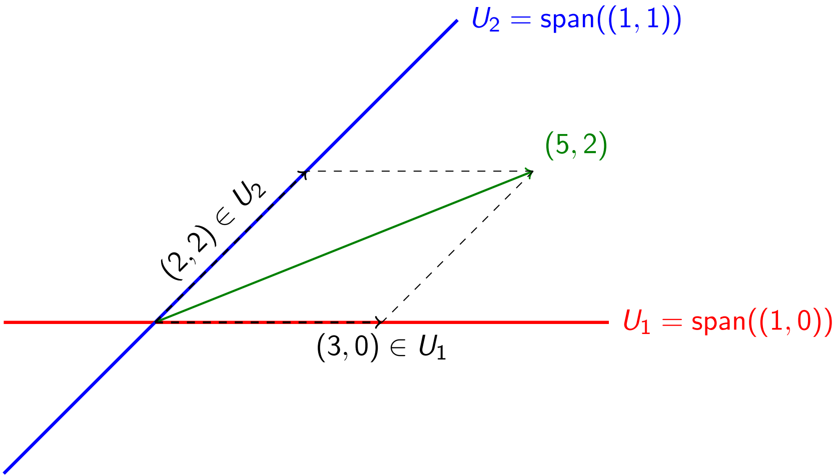
then we write $V = U_1 \oplus U_2$.
With respect to this direct sum we can project vectors onto subspaces, if $v = u_1 + u_2$ we write

$$P_{U_1}(v) = u_1 \quad \text{and} \quad P_{U_2}(v) = u_2.$$


$U_2 = \text{span}((1,1))$
$U_1 = \text{span}((1,0))$


$U_2 = \text{span}((1,1))$
$(5,2)$
$U_1 = \text{span}((1,0))$



In this example we have $P_{U_1}(5,2) = (3,0)$ and $P_{U_2}(5,2) = (2,2)$.

Theorem

Let U_1, U_2 be subspaces of V. Then $V = U_1 \oplus U_2$ if and only if

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Example

Consider the vector space $\mathrm{Mat}_n(\mathbb{R})$ and consider the subspaces
$\mathcal{S} = \{A \in \mathrm{Mat}_n(\mathbb{R}) | A^T = A\}$ (symmetric matrices)
$\mathcal{S}' = \{A \in \mathrm{Mat}_n(\mathbb{R}) | A^T = -A\}$ (skew-symmetric matrices)
Is $\mathrm{Mat}_n(\mathbb{R}) = \mathcal{S} \oplus \mathcal{S}'$?

External direct sum

Definition

Let V and W be vector spaces over the same field $\mathbb{F}$. We define the **external direct sum** as the $\mathbb{F}$-vector space

$$V \oplus W = V \times W = \{(v, w) \mid v \in V, w \in W\}$$

equipped with the natural addition and scalar multiplication:

$$(v, w) + (v', w') := (v + v', w + w') \qquad \lambda(v, w) = (\lambda v, \lambda w).$$

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Note that $\dim(V \oplus W) = \dim(V) + \dim(W)$.

Part II

Quotient spaces

Affine subsets

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Definition

Let U be a subspace of a vector space V. The set of all affine subsets

$$V/U := \{v + U \mid v \in V\}$$

is also called the **quotient space** of V by U. V/U is a vector space under the operations

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Note that $\dim(V/U) = \dim(V) - \dim(U)$.

Part III

Linear maps

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Let V and W be vector spaces over the same field $\mathbb{F}$. A map $F : V \to W$ is called **linear** if

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Let $\mathcal{B} = (v_1, \ldots, v_n)$ be a basis for V, let $\mathcal{B}' = (w_1, \ldots, w_m)$ be a basis for W. Then

$$F(\lambda_1 v_1 + \cdots \lambda_n v_n) = \lambda_1 F(v_1) + \cdots \lambda_n F(v_n).$$

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Express each $F(v_i)$ in the basis $\mathcal{B}'$ and put these coordinate vectors as columns in a matrix:

$$A = [F]_{\mathcal{B}',\mathcal{B}} = \begin{pmatrix} | & | & & | \\ F(v_1) & F(v_2) & \cdots & F(v_n) \\ | & | & & | \end{pmatrix} \in \mathrm{Mat}_{m\times n}(\mathbb{F})$$

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Then $F(v)_{\mathcal{B}'} = Av_{\mathcal{B}}$, so A is the matrix of F with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$.

Kernel and image

Definition

Let $F : V \to W$ be a linear map (or matrix). We define

- $\mathrm{Ker}(F) = \{v \in V \mid F(v) = 0\}$, the **kernel** or **nullspace** of F.
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Rank nullity theorem (dimension theorem)

$$\dim(\mathrm{Ker}(F)) + \dim(\mathrm{Im}(F)) = \dim(V).$$

$\dim(\mathrm{Im}(F))$ is called the **rank** of F.

Inverses

The **inverse** to a linear map $F : V \to W$ is a linear map $G : W \to V$ such that

$$G(F(v)) = v \text{ for all } v \in V \text{ and } F(G(w)) = w \text{ for all } w \in W.$$

We write $G = F^{-1}$ if such a map exists.

Part IV

Matrices

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Standard basis: $e_{ij} \in \mathrm{Mat}_{m\times n}(\mathbb{F})$ - matrix with a single 1 in position (i, j), for example:

$$\begin{pmatrix} 0 & 2 & 5 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix} = 2e_{12} + 5e_{13} - 3e_{23}.$$

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For a matrix $A = (a_{ij})_{ij} = \sum_{i,j} a_{ij}e_{ij}$ we define

- $A^T = (a_{ji})_{ij}$ the **transpose** of A
- $A^* = (\overline{a_{ji}})_{ij}$ the **Hermitian conjugate** (conjugate-transpose) of A (when $\mathbb{F} = \mathbb{C}$)

Matrix properties

A matrix $A = (a_{ij}) \in \mathrm{Mat}_{m\times n}(\mathbb{F})$ is called

- **Diagonal** if $a_{ij} = 0$ whenever $i \neq j$
- **Upper triangular** if $a_{ij} = 0$ whenever $i > j$
  (strictly upper triangular if $a_{ii} = 0$ also)
- **Lower triangular** if $a_{ij} = 0$ whenever $j > i$
  (strictly lower triangular if $a_{ii} = 0$ also)
- **Symmetric** if $a_{ij} = a_{ji}$
  (skew-symmetric if $a_{ij} = -a_{ji}$)
- **Hermitian** $a_{ij} = \overline{a_{ji}}$ (when $\mathbb{F} = \mathbb{C}$)
  (skew-Hermitian if $a_{ij} = -\overline{a_{ji}}$)

Multiplication

The standard basis-matrices multiply as

$$e_{ij} e_{kl} = \begin{cases} e_{il} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases} = \delta_{jk} e_{il}.$$

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Let $A = (a_{ij}) = \sum_{i,j} a_{ij}e_{ij} \in \mathrm{Mat}_{m\times n}(\mathbb{F})$ and let $B = (b_{ij}) = \sum_{i,j} b_{ij}e_{ij} \in \mathrm{Mat}_{n\times k}(\mathbb{F})$. Then

$$AB = \Big(\sum_{r=1}^{n} a_{ir}b_{rj}\Big)_{ij} \in \mathrm{Mat}_{m\times k}(\mathbb{F}).$$

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For example, with $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ we have

$$\mathrm{tr}(AB) = \mathrm{tr}\begin{pmatrix} 2 & 3 \\ 2 & 5 \end{pmatrix} = 7 \text{ and } \mathrm{tr}(BA) = \mathrm{tr}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 4 & 3 \end{pmatrix} = 7.$$

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It follows that $\mathrm{tr}(S^{-1}AS) = \mathrm{tr}(SS^{-1}A) = \mathrm{tr}(A)$, so the trace is basis-independent.

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$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A^4 = 0$$

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Similarly, a linear map $F : V \to V$ is called nilpotent if $F^n = 0$ for some n.

Block matrices

It is sometimes useful to consider a matrix as a matrix with matrix-coefficients:

$$X = \left(\begin{array}{cc|cc} 3 & 2 & 3 & 0 \\ 1 & 1 & 0 & 3 \\ \hline 0 & 0 & 6 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) = \left(\begin{array}{c|c} A & 3I \\ \hline 0 & 2A \end{array}\right) \quad \text{where} \quad A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.$$

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This can simplify calculations, for example:

$$X^2 = \left(\begin{array}{c|c} A & 3I \\ \hline 0 & 2A \end{array}\right)\left(\begin{array}{c|c} A & 3I \\ \hline 0 & 2A \end{array}\right) = \left(\begin{array}{c|c} A \cdot A + 3I \cdot 0 & A \cdot 3I + 3I \cdot 2A \\ \hline 0 \cdot A + 2A \cdot 0 & 0 \cdot 3I + 2A \cdot 2A \end{array}\right)$$

$$= \left(\begin{array}{c|c} A^2 & 9A \\ \hline 0 & 4A^2 \end{array}\right) = \left(\begin{array}{cc|cc} 11 & 8 & 27 & 18 \\ 4 & 3 & 9 & 9 \\ \hline 0 & 0 & 44 & 32 \\ 0 & 0 & 16 & 12 \end{array}\right).$$

Part V

Echelon forms

Matrices in echelon forms

$$A = \begin{pmatrix} \textcircled{1} & 1 & 9 & 2 & 1 & 8 \\ 0 & \textcircled{2} & 4 & 1 & 2 & 5 \\ 0 & 0 & 0 & \textcircled{4} & 1 & 9 \\ 0 & 0 & 0 & 0 & \textcircled{3} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \textcircled{1} & 0 & 7 & 0 & 0 & 2 \\ 0 & \textcircled{1} & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 0 & 2 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Note that $\mathrm{Ker}(A) = \mathrm{Ker}(B)$ when the matrices are row equivalent.

Part VI

Elementary matrices

Row operations as matrix products

$$E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

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Multiplying the first row of A by 3.

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Adding (-2) times the first row to the second row of A.

$$E_2A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 & -3 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Multiplying the first row of A by 3.

$$E_3A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

Row operations as matrix products

$$E_1A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

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Switching rows 2 and 3 of A.

Elementary matrices

Elementary matrix	Corresponding row operation
$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & \lambda & \ddots & \\ & & & & 1 \end{pmatrix} = I + \lambda e_{ij}$ (λ in position (i, j))	Add λ times row j to row i

Elementary matrices

Elementary matrix	Corresponding row operation
$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = I + (\lambda - 1)e_{ii}$ (identity except $\lambda \neq 0$ on position (i, i))	Multiply row i by a nonzero scalar λ

Elementary matrices

Elementary matrix	Corresponding row operation
$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & & 1 & & \\ & & & & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & & & \\ & & & 1 & & 0 & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix}$ $= I - e_{ii} - e_{jj} + e_{ij} + e_{ji}$ (I but with rows i and j switched)	Switching rows i and j

Products of elementary matrices

Note that all elementary matrices are invertible, and the inverse is also an invertible matrix.

Theorem

Every invertible matrix is the product of elementary matrices.

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Theorem

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Proof: Perform row operations on A until it becomes the identity matrix. If the row operations correspond to elementary matrices $E_1, \dots, E_n$ we get

$$E_n \cdots E_2 E_1 A = I \quad \Leftrightarrow \quad A = E_1^{-1} E_2^{-1} \cdots E_n^{-1}.$$

Part VII

LU-decompostion

LU-decomposition

Definition

An **LU-decomposition**, or an **LU-factorization** of $A \in \mathrm{Mat}_{m \times n}(\mathbb{F})$ is a factorization

$$A = LU$$

where

- L is lower triangular $m \times m$-matrix
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Example:

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & 7 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

The algorithm

Example

Find an LU factorization of

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & -1 \\ -2 & 2 & -1 & 5 \end{pmatrix}.$$

The algorithm

Example

Find an LU factorization of

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Idea: Reduce A to REF by standard row operations, the resulting matrix is upper triangular:

$$E_n \cdots E_2 E_1 A = U \Leftrightarrow A = LU = (E_n \cdots E_2 E_1)^{-1} U,$$

where $L = (E_n \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_n^{-1}$ is lower triangular since we only add higher rows to lower rows to achieve the REF.

LDU-decomposition

Definition

An **LDU-decomposition**, or an **LDU-factorization** of $A \in \mathrm{Mat}_{m\times n}(\mathbb{F})$ is a factorization

$$A = LDU$$

- L is lower triangular $m \times m$-matrix with ones on the diagonal
- D is a diagonal $m \times m$-matrix
- U is an upper triangular $m \times n$-matrix with ones on the diagonal

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This can be obtained from the LU-factorization by factoring $U = DU'$ by dividing out the leading coefficients of each row of U.

Application

Suppose that we want to solve a large linear system $Ax = b$ many times for different right sides b.

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So we can solve two triangular systems instead with back-substitution, this is significantly faster for large matrices A.