Assignment I

Honours Linear Algebra · TATA53 · Spring 2025 · Jonathan Nilsson

Instructions: Write your name, personal number, and program (if applicable) on the first page. The assignment should be solved individually. Your solutions should not depend on digital tools (unless otherwise stated). Write complete solutions with clear answers. You are allowed to discuss the problems with other students, but you should write your solutions individually, don't write anything that you don't understand yourself. Write your solutions on paper by hand and staple them together. Some problems marked ⚔ are harder, it is not necessary to solve all the problems. You should be prepared to present your solutions on *one* of the two associated seminars. To pass the course you need at least 60% on each assignment. Your final grade is based on the assignment average, grade 3/4/5 correspond to 60%/75%/85%.
Deadline: 15.00 on February 14 2025. Hand in the assignments in the compartment labelled TATA53-submissions one floor up in the B-building just above entrance 21, or bring them to the *first* seminar starting 15 minutes later.

1. Find the *reduced* row echelon form (RREF) of the matrix A below. Use your result to find a basis for $\ker(A)$. [6]

$$A = \begin{pmatrix} 1 & i & 2 & 2i \\ 2i & -2 & 2+i & 1-i \\ 2-3i & 3+2i & 1-i & -2+6i \end{pmatrix}$$

2. Let $\mathcal{P}_3$ be the real vector space of polynomials of degree ≤ 3. Consider the subsets: [7]

$$\begin{aligned} U_1 &= \{p(x) \in \mathcal{P}_3 \mid p(-1) = 0 = p(3)\}, \\ U_2 &= \{p(x) \in \mathcal{P}_3 \mid p(x) = p(2-x)\}, \\ U_3 &= \{p(x) \in \mathcal{P}_3 \mid p''(x) = 0\}, \\ U_4 &= \{p(x) \in \mathcal{P}_3 \mid p(5) = 4\}. \end{aligned}$$

   (a) One of the four subsets is not a *subspace*, which one?

   (b) For each of the three subspaces above, find a basis for it.
   (No motivation is needed here, just an answer)

   (c) $\mathcal{P}_3$ can be written as a direct sum of two of these subspaces, which ones?

   (d) With respect to the direct sum in (c), find the projections of $p(x) = x^3$ onto each of the two subspaces.

3. Consider $V = \mathbb{R}^2$ as just a set. V becomes a real vector space under the following definitions of addition and multiplication by real numbers: [6]

$$(x, y) + (x', y') := (x + x' - 1, y + y')$$

$$\lambda \cdot (x, y) := (\lambda x - \lambda + 1, \lambda y)$$

   (a) What is the zero-vector (the additive identity) of the vector space V?

   (b) Find $-(2, 1)$, the additive inverse of the vector $(2, 1)$ in V.

   (c) Let $e_1 = (3, -2)$ and $e_2 = (0, 1)$. Is (e_1, e_2) a basis for V?

4. Determine whether each statement below is true or false. Give a very short proof or a counterexample to each statement. [8]

   (a) If $F : U \to V$ and $G : V \to W$ are linear maps, then their composition $G \circ F$ is also a linear map.

   (b) The determinant function $\det : \mathrm{Mat}_2(\mathbb{R}) \to \mathbb{R}$ is a linear map.

   (c) For $U = \mathrm{span}\big((1,1,0),(0,1,1)\big)$ we have $(1,2,3) + U = (2,5,5) + U$ in $\mathbb{R}^3/U$.

   (d) ⚔ Let $\mathbb{F}$ be a field with finitely many elements. Then for every invertible matrix $A \in \mathrm{Mat}_n(\mathbb{F})$ there exists $m > 0$ such that $A^m = I$.

5. Find LU- and LDU-factorizations of [5]

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 0 & -1 \\ -1 & 2 & -1 & 0 \end{pmatrix} \in \mathrm{Mat}_{3\times 4}(\mathbb{R}).$$

6. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F(v) = (3,4,0) \times v$, the usual vector product on $\mathbb{R}^3$. [6]

   (a) Show that F is linear and find the matrix of F with respect to the standard basis of $\mathbb{R}^3$.

   (b) Find all eigenvalues and eigenvectors of F. Is F diagonalizable?

   (c) The matrix from (a) can be viewed as a map $G : \mathbb{C}^3 \to \mathbb{C}^3$. Find all eigenvalues and eigenvectors of G. Is G diagonalizable?

7. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_3(\mathbb{Z}_3)$. [6]

   (a) Find all eigenvalues and eigenvectors of A, and determine if A is diagonalizable.

   (b) Find $A^{-1} \in \mathrm{Mat}_3(\mathbb{Z}_3)$.

   (c) Compute A^{1000}.

8. There are six buttons in a row, each with a light that is either on or off. When you press a button, its light switches on↔off, but it also switches the light on the adjacent buttons. At a given time, the status of the lights can be identified with a vector in $(\mathbb{Z}_2)^6$, where 1 means on and 0 means off. Pressing a button then corresponds to adding a certain vector to the status-vector, for example, pressing the third button corresponds to adding $(0,1,1,1,0,0)$. To solve the puzzle all the lights should be turned on. [6]

   (a) Suppose the light-status is currently $(0,0,0,1,0,1)$. What buttons should be pressed to solve the puzzle?

   (b) ⚔ Show that the puzzle is solvable regardless of the starting position.

   (c) ⚔ Now consider the same puzzle but with n lights in a row. For what values of n is every starting position solvable?