Assignment II

Honours Linear Algebra TATA53 Spring 2025 Jonathan Nilsson

Instructions: Write your name, personal number, and program (if applicable) on the first page. The assignment should be solved individually. You are allowed to discuss the problems with other students, but write your own solutions and don't write anything that you don't understand yourself. Your solutions should not depend on digital tools. Write complete solutions with clear answers. It is good to be concise, but include enough motivation so that an average course participant can follow and be convinced by all the steps of your argument. Write your solutions on paper by hand and staple them together. It is ok to solve several problems on the same page. Some problems marked ⚔ may be harder, it is not necessary to solve all the problems. You should be prepared to present your solutions on *one* of the two associated seminars. To pass the course you need at least 60% on each assignment. Your final grade is based on the assignment average, grade 3/4/5 correspond to 60%/75%/85%.
Deadline: 10.00 on March 10 2025. Hand in the assignments in the compartment labelled TATA53-submissions one floor up in the B-building just above entrance 21, or bring them to the *first* seminar starting 15 minutes later.

1. Find A^{20} and e^B and $\sin(C)$ where [3]

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} \frac{\pi}{3} & 1 & 0 \\ 0 & \frac{\pi}{3} & 1 \\ 0 & 0 & \frac{\pi}{3} \end{pmatrix}.$$

2. For a complex matrix A we know that its characteristic and minimal polynomials are [5]

$$p_A(t) = t^7(t-6)^7 \quad \text{and} \quad m_A(t) = t^3(t-6)^4.$$

We also know that $\operatorname{rank}(A) = 11$ and $\dim\ker(A^2) = 6$, and that the geometric multiplicity is the same for both eigenvalues. Find the Jordan form of A.

3. We define the *Manhattan norm* of a vector $(x, y) \in \mathbb{R}^2$ as $\|(x, y)\|_{\mathrm{Mh}} := |x| + |y|$. [5]

   (a) Verify that $\|\cdot\|_{\mathrm{Mh}}$ satisfies the triangle inequality for norms.

   (b) Show that the Manhattan norm can not be derived from an inner product: there is no inner product $\langle \cdot, \cdot \rangle$ for which $\|v\|_{\mathrm{Mh}} = \sqrt{\langle v, v \rangle}$ for all $v \in \mathbb{R}^2$.
   Hint: Show that the Manhattan norm does not satisfy the parallelogram law.

   (c) ⚔ With respect to the Manhattan norm, find every point in $\mathbb{R}^2$ that lies equally close to the points $(1, 0)$ and $(0, 3)$, sketch your result.

4. Jordanize the matrix A below. In other words, find a matrix S and a matrix J in Jordan form such that $A = SJS^{-1}$. [8]

$$A = \begin{pmatrix} -2 & -3 & 4 & 5 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 3 & 0 \\ -4 & -3 & 4 & 7 \end{pmatrix}$$

5. Let $\mathcal{P}$ be the real vector space of polynomials with real coefficients. Define an inner product on $\mathcal{P}$ by [6]

$$\langle p, q\rangle := \int_0^\infty p(x)q(x)e^{-x}dx.$$

(a) Verify that $\langle p, p\rangle \geq 0$ with equality if and only if $p = 0$.

(b) Find the angle between the polynomials 1 and x.

(c) Derive an explicit formula for $\langle x^m, x^n\rangle$.

6. Use the methods of this course to solve the following boundary value problem: [7]

$$\begin{cases} x_1'(t) = \ \ 7x_1(t) + 4x_2(t) \\ x_2'(t) = -x_1(t) + 3x_2(t) \end{cases} \quad \text{with} \quad x_1(0) = 1 = x_2(0).$$

7. In *Lie theory*, the exponential map allows us to connect *Lie groups* and *Lie algebras*. Let $A \in \mathrm{Mat}_n(\mathbb{R})$ be a skew-symmetric matrix. [5]

(a) Prove that e^A is an orthogonal matrix.

(b) Now let $n = 2$, so that $A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$ for some $\theta \in \mathbb{R}$. Compute e^A explicitly.

You may use the results of Example 3.1.7 in the compendium without proof.

8. Give an example of a matrix $A \in \mathrm{Mat}_2(\mathbb{Z}_5)$ which is impossible to Jordanize; it should not be possible to express $A = SJS^{-1}$ where all matrices have coefficients in $\mathbb{Z}_5$ and J is in Jordan form. [3]

9. Let F be the operator on $\mathrm{Mat}_n(\mathbb{R})$ given by $F(A) = 2A + 3A^T$. Find the minimal polynomial of F. [3]

10. Let F_n be the operator on a $(2^n - 1)$-dimensional vector space where $F_n(e_1) = 0$ and $F_n(e_k) = e_{\lfloor \frac{k}{2} \rfloor}$ for $k > 1$. Find the Jordan form of F_n for each $n \geq 1$. [5]

$$\begin{tikzcd}
e_4 \arrow[dr, mapsto] & & \\
& e_2 \arrow[ddr, mapsto] & \\
e_5 \arrow[ur, mapsto] & & \\
& & e_1 \arrow[r, mapsto] & 0 \\
e_6 \arrow[dr, mapsto] & & \\
& e_3 \arrow[uur, mapsto] & \\
e_7 \arrow[ur, mapsto] & &
\end{tikzcd}$$

Illustration for the case $n = 3$