Assignment IV

Honours Linear Algebra · TATA53 · Spring 2025 · Jonathan Nilsson

Instructions: Write your name, personal number, and program (if applicable) on the first page. The assignment should be solved individually. Your solutions should not depend on digital tools, except for problems marked by ⌨. Write complete solutions with clear answers. It is good to be concise, but include enough motivation so that an average course participant can follow and be convinced by all the steps of your argument. It is ok to write the solutions to several problems on the same page. You are allowed to discuss the problems with other students, but you should write your solutions individually, don't write anything that you don't understand yourself. Write your solutions on paper by hand and staple them together. Some problems marked ⚔ are harder, it is not necessary to solve all the problems. You should be prepared to present your solutions on *one* of the two associated seminars.

Deadline: 10.00AM on May 12 2025. Hand in the assignments in the compartment labelled "TATA53 assignment submissions" one floor up in the B-building just above entrance 21, or bring them to the *first* seminar starting 15 minutes later.

1. Find the compact singular value decomposition $A = \tilde{U}\tilde{\Sigma}\tilde{V}^*$ of the matrix A below. Also find the matrix of rank 1 which is closest to A with respect to the Frobenius norm. [6]

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

2. Find the line in $\mathbb{R}^2$ that lies closest to the following 6 points (x, y) in the *total* least squares sense. [6]

x	6	2	4	7	5	6
y	4	3	2	3	1	5

3. For a 4×4-matrix A we know the Frobenius norm, the condition number, the nuclear norm, and the operator norm of the inverse of A: [4]

$$\|A\|_F = \sqrt{153}, \quad \kappa(A) = \frac{5}{3}, \quad \|A\|_{☢} = 13 + 8\sqrt{2}, \quad \left\|A^{-1}\right\|_{\text{op}} = \frac{\sqrt{2}}{6}.$$

Find the matrix Σ in the singular value decomposition $A = U\Sigma V^*$ of A.

4. Consider the space $\mathcal{P}_2$ of polynomials of degree ≤ 2 with real coefficients, it has a basis $(e_1, e_2, e_3) = (1, x, x^2)$. The map $f : \mathcal{P}_2 \to \mathbb{R}$ given by $f(p(x)) = \int_{-1}^{2} p(x)dx$ is a linear functional on $\mathcal{P}_2$. Express f in the dual basis (e_1^*, e_2^*, e_3^*). [4]

5. Find the polar factorization $A = UP$ of the matrix $A = \begin{pmatrix} 3 & 2 \\ -1 & 6 \end{pmatrix}$ where U is unitary and P is positive definite. Also determine the condition number $\kappa(A)$. [4]

6. Determine whether each statement below is true or false. Motivate with a short proof or a counter-example. All matrices below are square of size ≥ 2. [8]

   (a) If σ is a singular value of A, then σ^2 is a singular value of A^2.

   (b) Defining $\|A\| := \sigma_2$ (the second singular value of A) gives a norm on $\mathrm{Mat}_n(\mathbb{C})$.

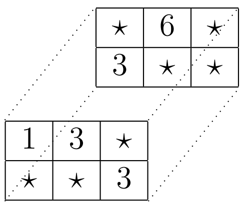
   (c) Given a polar-factorization $A = UP$, we have $\kappa(A) = \kappa(P)$.

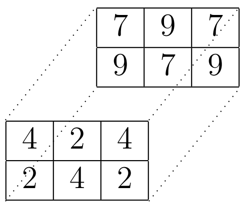
   (d) If A and B are invertible, then so is their Kronecker product $A \otimes B$.

7. 3d-arrays of the form below correspond to vectors in $\mathbb{R}^2 \otimes \mathbb{R}^3 \otimes \mathbb{R}^2$. [4]

   (a) Complete the left array so that it corresponds to a *pure* tensor.

   (b) Write the right array as a sum of pure tensors, with as few terms as possible. *Just an answer is needed here.*





8. Words that appear together tend to be related. This is the idea behind latent semantic analysis; by analyzing word frequencies we can infer the meaning or context of the words. We go through 12 poems and count the number of occurrences 8 specific words. We find the following word frequencies across the 12 poems. [8]

Poem title	king	queen	throne	sword	battle	love	beauty	rose
Epic of Kings	5	1	3	4	4	0	0	0
Royal Chronicles	4	2	5	3	3	1	0	0
Sword and Crown	3	0	2	5	5	0	0	0
Love in War	1	1	0	2	2	5	3	1
Tales of Queens	0	5	3	1	1	2	2	1
The Rose and the Blade	2	2	1	4	4	2	1	1
Beauty and Blood	0	1	0	2	3	3	5	2
Throne of Hearts	2	2	4	2	2	3	2	1
War and Roses	1	1	1	3	4	4	2	3
Battle Hymn	0	0	0	4	5	1	1	0
Kingdom of Love	3	2	1	2	1	4	2	1
Royal Blood	4	3	3	3	2	1	0	0

Write a computer program that performs a singular value decomposition of this data-matrix. Plot the 8 words as points in a 2d-diagram by projecting the words onto the first two singular vectors. Briefly discuss your results, and comments on the clusters that appear. Print and attach your code and its output.

9. Derive the Clebsch-Gordan decomposition formula for linear maps: find a general formula for the Jordan form of the Kronecker product [6]

$$J_m(\lambda) \otimes J_n(\mu),$$

where λ and μ are nonzero. *Hint: start with the case $\lambda = \mu = 1$ and $m = 2$.*