Exercises for TATA55, batch 2, 2018

November 29, 2018

Solutions to the exercises below should be handed in no later than November 30.

1. (3p) Suppose that G is a group, A, B are subgroups of G, and that $g \in G$. Show that $gA \cap gB$ is a left coset in $A \cap B$.

2. (3p) Determine subgroups K, H in D_4 such that
$$\{1\} \triangleleft K \triangleleft H \triangleleft D_4$$
with all inclusions proper. Determine D_4/K and $(D_4/K)/(H/K)$.

3. (5p) Let G be a group.
   (a) Suppose that $S \subseteq G$ is a subset of G such that $gsg^{-1} \in S$ for all $g \in G$ and all $s \in S$. Show that $\langle S \rangle$, the subgroup generated by S, is normal in G.
   (b) Put $K = \langle \{ xyx^{-1}y^{-1} | x, y \in G \} \rangle$. Show that $K \triangleleft G$.
   (c) Show that G/K is abelian.
   (d) If $N \triangleleft G$ and G/N is abelian, show that $K \subseteq N$.
   (e) If $K \subseteq H \leq G$, show that $H \triangleleft G$.

4. (3p) Let $G \subseteq S_{\mathbb{R}}$ be given by all affine maps $\phi_{a,b}$, $a, b \in \mathbb{R}$, $a \neq 0$, $\phi_{a,b}(x) = ax + b$.
   (a) Show that G is a subgroup. Is it normal?
   (b) Let $N = \{ \phi_{1,b} | b \in \mathbb{R} \}$. Show that $N \triangleleft G$.
   (c) Determine G/N.

5. (4p) Let $[5] = \{1, 2, 3, 4, 5\}$, and let $X = \binom{[5]}{3}$, the set of unordered triplets of [5].
   (a) S_5 acts naturally on [5]. Show that the induced action $\phi.\{a, b, c\} = \{\phi(a), \phi(b), \phi(c)\}$ indeed determines an action of S_5 on X.
   (b) Determine the number of orbits of this action.
   (c) Let $H = \langle (1, 2, 3, 4, 5) \rangle$ act on X as above. Determine the number of orbits.

(d) Same question for $K = \langle (1,2) \rangle$.

(e) Partial credits if you solve the above questions for S_4 acting on $\binom{[4]}{2}$ instead.

6. (5p) Show that the number of conjugacy classes in a finite group G is given by

$$\frac{1}{|G|} \sum_{g \in G} |C_G(g)|, \qquad C_G(g) = \{ h \in G | gh = hg \}.$$

Determine the number of conjugacy classes in D_8 and D_9.

7. (1p+2p) Let $\mathbf{u} = (u_1, u_2)^t$ and $\mathbf{v} = (v_1, v_2)^t$ be two linearly independent vectors in $\mathbb{R}^2$, and let $B = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$. Put $L = \{ a\mathbf{u} + b\mathbf{v} | a, b \in \mathbb{Z} \}$. This is called the lattice spanned by $\mathbf{u}$ and $\mathbf{v}$.

(a) Show that $L \leq \mathbb{R}^2$, and that $\mathbb{R}^2/L \simeq (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$.

(b) If $\mathbf{f}, \mathbf{g}$ are two other linearly independent vectors in $\mathbb{R}^2$, with associated lattice M and matrix C, show that $L = M$ if and only if $B = CU$ for some two-by-two matrix U with integral entries, and determinant ± 1.

8. (4p) Denote by K the hypercube $K = \{ (x_1, x_2, x_3, x_4) | 0 \leq x_1, x_2, x_3, x_4 \leq 1 \}$, and let $V = \{ (x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in \{0, 1\} \}$ be the set of its vertices. Let $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$, $\mathbf{e}_4 = (0, 0, 0, 1)$. Let $\Delta = \text{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)$.

Let $\sigma \in S_4$ act on K by $\sigma.(x_1, x_2, x_3, x_4) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$.

(a) What are the sizes of the orbits?

(b) Put $\Delta_\sigma = \{ \sigma.(x_1, x_2, x_3, x_4) | (x_1, x_2, x_3, x_4) \in \Delta \}$. Determine the volume of this simplex, and show that

$$K = \cup_{\sigma \in S_4} \Delta_\sigma,$$

with $\Delta_\sigma \cap \Delta_\tau$ a simplex of dimension < 4, hence of volume zero, for $\sigma \neq \tau$.

(c) Partial credit if you solve the corresponding questions for $n = 3$, even more partial if you look at $n = 2$.