Exercises for TATA55, batch 2, 2019

October 11, 2019

Solutions to the exercises below should be handed in no later than XXXber XXX, 2019.

1. (3p) Let $H = \{(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \subset S_4$. Show that H is normal in S_4, and describe S_4/H.

2. (3p) Count the number of permutations in S_n with cycle decomposition consisting of c_1 fixed points, c_2 cycles of length two, and so forth.

3. (1p+1p+3p) The center of a group G is

$$Z(G) = \{ g \in G \,|\, gx = xg \text{ for all } x \in G \}.$$

   (a) Show that $Z(G) \leq G$
   (b) Show that $Z(G) \lhd G$
   (c) Find $Z(D_n)$

4. (1p+1p+1p+3p) Recall that for a group G, two elements $g_1, g_2 \in G$ are conjugate if $g_2 = hg_1h^{-1}$ for some $h \in G$; this is an equivalence relation, and the classes are called conjugacy classes.

   (a) Show that $H \lhd G$ if and only if it is a union of conjugacy classes
   (b) If $H \leq G$ and h_1, h_2 are conjugate in H, show that they are conjugate in G
   (c) Need the converse hold? What if $H \lhd G$?
   (d) Find the conjugacy classes of the alternating group $A_4 \lhd S_4$, using the theorem that says that permutations in S_n are conjugate iff they have the same cycle type.

5. (3p) Let G, H, and K be finitely generated abelian groups. Show that if $G \times H \simeq G \times K$, then $H \simeq K$. Give a counterexample to show that this cannot be true in general.

6. (3p) Detective Duncan has the following clues to the perpetrator of a particularly heinous crime (fomenting torsionist tendencies):

   (a) The "perp" is a finite abelian group
   (b) The size of the "perp" is 200000
   (c) The maximal order of an element of the "perp" is 200

   Help Duncan narrow down the list of suspects!