Exercises for TATA55, batch 3, 2019

November 10, 2019

Solutions to the exercises below should be handed in no later than XXXber XXX, 2019.

The first two exercises are from Judson; read Judson for terminology. To solve the last 5 exercises, you are encouraged to use a computer.

1. (3p) Let H be a subgroup of a group G. Prove or disprove that the normalizer of H is normal in G.

2. (4p) Let G be a group of order p^r, p prime. Prove that G contains a normal subgroup of order p^{r-1}.

3. (3p) The dihedral group D_n acts naturally on $[n] = \{1, 2, \dots, n\}$. For $n = 4, 5, 6, 7$, find
   (a) The stabilizers of each $i \in [n]$, are they all isomorphic?
   (b) The fixedpoints of all group elements
   (c) The number of ways of coloring a necklace with n beads, using k colors, up to dihedral symmetry.

4. (3p) The rotational symmetries of a cube has 24 elements. The group acts on the vertices of the cube.
   (a) Show that two rotations by 90 degrees suffice to generate the group
   (b) Find all stabilizers and show that they are isomorphic
   (c) Calculate the number of inequivalent ways of coloring the vertices using k colors

5. (3p) The full symmetry group of the cube (including reflections) has 48 elements
   (a) Show that the two rotations by 90 degrees, together with the antipodal map, generates this larger group
   (b) Can it be generated by fewer elements?
   (c) In how many inequivalent ways can we now color the vertices, using k colors?

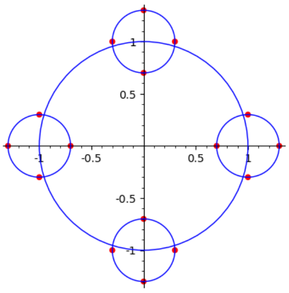
6. (3p) Let G be an undirected graph whose vertex set is a subset of $[n]$. Then its edge set is a subset of $\binom{[n]}{2}$. The symmetric group S_n acts on $[n]$, and in a natural way on $\binom{[n]}{2}$, and finally on $\mathcal{P}\left(\binom{[n]}{2}\right)$, thus on the set of all undirected graphs on $[n]$.

   (a) How? And why do the orbits correspond to isomorphism classes of graphs?

   (b) For $n = 3$, find the number of non-isomorphic graphs, and give a representative from each class. Are there non-isomorphic graphs with the same number of edges?

   (c) Do the same for $n = 4, 5$.

7. (3p) Consider this arrangement of 16 points:



Enumerate the points using 1,2,3,4 for the rightmost small circle, et cetera. There is a strange group acting on the points, in the following way:

   (i) We can rotate the smaller circles, letting the points tag along without turning

   (ii) We can rotate all small circle counter-clockwise one quarter turn, then the last three small circles, then the last two, then the last

Call the first operation the "megaturn" and the second operation the "intricate turn". The "strange group" is generated by these, and acts on $[16]$, so we regard it as a subgroup of S_{16}.

   (a) What is the size of our strange group? Is it abelian?

   (b) Find the stabilizer of point 1. Then express all elements in the stabilizer in terms of intricate and/or mega turns.

   (c) In how many inequivalent ways can we color the 16 points, if colorings that can be transformed into each other using intricate and/or mega turns are considered equivalent?