Linnæus University
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Master Thesis

Polya's Enumeration Theorem

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Abstract

Polya's theorem can be used to enumerate objects under permutation groups. Using group theory, combinatorics and some examples, Polya's theorem and Burnside's lemma are derived. The examples used are a square, pentagon, hexagon and heptagon under their respective dihedral groups. Generalization using more permutations and applications to graph theory.

Using Polya's Enumeration theorem, Harary and Palmer [5] give a function which gives the number of unlabeled graphs *n* vertices and *m* edges. We present their work and the necessary background knowledge.

Key-words: Generating function; Cycle index; Euler's totient function; Unlabeled graph; Cycle structure; Non-isomorphic graph.

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Contents

1 Introduction

Some of the most difficult problems in mathematics involve counting. There are several reasons for this difficulty, some of which are technical and others more conceptual. A frequently encountered technical difficulty is that the objects to be counted may not be sequentially arranged (e.g., the school bus packed with jelly beans). A common conceptual difficulty occurs when different objects are identified (considered to be equal) for enumeration purposes (e.g., it may only be of interest to know how many colors of jelly beans are in the bus). In modern language, there may be an equivalence relation imposed on the objects. The problem then is to enumerate equivalence classes.

Consider the following situation. Suppose D is a set of m objects which, for simplification, we take to be $\{1,2,...,m\}$. Suppose further that we wish to color the objects in D and that we have at our disposal a set C of k colors, $C = \{c_1, c_2, ..., c_k\}$. We may think of a coloring of D as a function $f : D \rightarrow \{c_1, c_2, ..., c_k\}$. Let c^m denote the set of all such colorings.

So far, so good. But now we introduce the confusion of equivalent colorings. Suppose G is a group of permutations on D[1]. We say that two colorings f_1 and f_2 of D are equivalent $(mod\ G)$ if there is a permutation $\sigma \in G$ such that $f_1\sigma = f_2$. This equivalence relation imposes a partion on the set of colorings of D. The equivalent classes so obtained are called color patterns. The question then becomes: How many color patterns are there? A very general and elegant theorem [2] due to George Polya supplies the answer.

The main aim of the thesis is to describe the enumeration method bases on Polya's Enumeration Theorem (PET). By using this method to compute the number of colorings of geometric objects and non-isomorphic graphs.

The problem of counting the number of non-isomorphic graphs of a given order n is perhaps one of the most obvious problems in any study of graphs. When the vertices are labeled the answer is readily obtained as $2^{\binom{n}{2}}$, since each of the $\binom{n}{2}$ possible edges may be either present or missing. On the other hand, when the vertices are unlabeled, the problem becomes more interesting. For small values of n it is easy to determine.

For instance, when $n = 2$, there are two graphs; when $n = 3$, there are four, and when $n = 4$, there are eleven distinct unlabeled graphs. The difficulty arises in determining how many graphs are truly distinct for larger values of n.

In 1927, the first solution to the problem appeared in [8]. Subsequently the problem was successfully solved by other mathematicians, including Polya, whose enumeration theorem proves valuable in our approach to the problem. Throughout this portion, definitions and theorems are taken from [7][4]. with the exception of the statement of Polya's theorem, which is taken from [9].

We start our thesis with the basic definitions in Sec 2 and 6, which are necessary to understand before the proof of Polya's Enumeration Theorem (PET) in Sec 3 and non-isomorphic graphs in Sec 7 respectively. In order to find the number of colorings of regular n-gons in Sec 5, we solve some examples of square and pentagon for different number of colors in Sec 4.

2 Preliminaries

In this section, we shall define some basic definitions with examples to prove the Polya's enumeration theorem.

2.1 Permutations

Let X be a set. By a permutation on X we mean a bijective mapping $\sigma : X \to X$. We will only study permutations in the case when the set X is finite.
A permutation on $X = \{a_1, a_2, ...a_m\}$ will then be denoted by a $2\times m$ matrix

$$\sigma = \begin{pmatrix} a_1 & a_2 & ... & a_m \\ \sigma(a_1) & \sigma(a_2) & ... & \sigma(a_m) \end{pmatrix}.$$

Since σ is bijective, each element on X is mapped onto exactly one element in X by σ. Therefore each element in X occures exactly once in the second row of the matrix above.

Example 2.1. Suppose $X = \{1,2,3\}$ and that $\sigma : X \to X$ is fulfills $\sigma(1) = 1$, $\sigma(2) = 3$, and $\sigma(3) = 2$. Then σ is a permutation on X, and we write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

Definition 2.1. Let ρ and σ be permutations on X. Then the **product** $\rho\sigma$ of ρ and σ is defined as the composite mapping $\rho \circ \sigma$. For each x$\in$ X we thus have

$$\rho\sigma(x) = \rho o \sigma(x) = \rho(\sigma(x)).$$

Example 2.2. Let

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \rho\sigma(1) &= \rho(\sigma(1)) = \rho(3) = 2 \\ \rho\sigma(2) &= \rho(\sigma(2)) = \rho(2) = 3 \\ \rho\sigma(3) &= \rho(\sigma(3)) = \rho(4) = 1 \\ \rho\sigma(4) &= \rho(\sigma(4)) = \rho(1) = 4. \end{aligned}$$

Hence

$$\rho\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}.$$

Example 2.3. The product $\sigma_1\sigma_2\sigma_3$, where

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, \text{ and } \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},$$

equals

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

Note that a product of permutations is read from right to left.

Let X be a set. Then we can form the set S_X of all permutations on X. Since the product of two permutations on X is again a permutation on X (the composition of two bijective mappings is a bijective mapping), multiplication of permutations is a binary operation on S_X. Moreover, composition of mappings is associative, whence multiplication of permutations is an associative binary operation. The identity mapping on X (i.e the mapping ε :X $\rightarrow$ X defined by $\varepsilon(x) = x$ for all x $\in$ X) is a permutation on X. This mapping is identity mapping with respect to multiplication of permutations

$$\sigma\varepsilon = \varepsilon\sigma = \sigma \text{ for all } \sigma \in S_X.$$

Since each permutation on X is a bijective mapping, it has an inverse, which is also a bijective mapping (i.e a permutation).

Theorem 2.1. *The set S_X is a group under multiplication of permutations.*

Definition 2.2. A group is called a **permutation group**, if all of its elements are permutations on a set X, and if its binary operation is multiplication of permutations.

Example 2.4. Consider the permutations

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

From example 2.1, their inverses are given by

$$\rho^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \text{ and } \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix},$$

respectively.

2.2 Group action

Definition 2.3. Let X be a set and G a group. An **action** of G on X is a map $* : G \times X \rightarrow$ X such that

1. ex = x for all x $\in$ X,
2. $(g_1g_2)(x) = g_1(g_2x)$ for all x $\in$ X and all $g_1, g_2 \in$ G.

Under these conditions X is a **G-set**.

Example 2.5. Let X be any set, and let H be a subgroup of the group S_x of all permutations of X. Then X is an H-set, where the action of $\sigma \in$ H on X is its action as an element of S_x, so that $\sigma x = \sigma(x)$ for all x $\in$ X. Condition 2 is a consequence of the definition of permutation multiplication as function composition, and condition 1 is immediate from the definition of the identity permutation as the identity function. Note that, in particular, $1, 2, 3, ..., n$ is an S_n set.

Each permutation σ of a set X determines a natural partition of X into cells with the property that a,b $\in$ X are in the same cell if and only if $b = \sigma^n(a)$ for some n $\in$ Z. We establish this partition using an appropriate equivalence relation:

$$\text{For } a, b \in X, \text{ let } a \sim b \text{ if and only if } b = \sigma^n(a) \text{ for some } n \in Z. \tag{2.1}$$

Theorem 2.2. *The relation $\sim$ defined by condition (2.1) is indeed an equivalence relation.*

Proof. Clearly a $\sim$ a since a = $\varepsilon(a) = \sigma^0(a)$. So, $\sim$ is reflexive. If a $\sim$ b, then b = $\sigma^n(a)$ for some n $\in$ Z. But then a = $\sigma^{-n}(b)$ and -n $\in$ Z, so b $\sim$ a. Which shows that $\sim$ is symmetric. Suppose a $\sim$ b and b $\sim$c, then $b = \sigma^n(a)$ and $c = \sigma^m(b)$ for some m,n $\in$ Z. Substituting, we find that c = $\sigma^m(\sigma^n(a)) = \sigma^{n+m}(a)$, so a $\sim$ c. That is $\sim$ is transitive.
Hence the result. □

Definition 2.4. Let σ be a permutation of a set X. The equivalence classes in X determined by the equivalence relation (2.1) are the **orbits** of σ.

Note: Since the identity permutation ε of X leaves each element of X fixed, the orbits of ε are the singletons of X.

Example 2.6. Find the orbits of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 8 & 6 & 7 & 4 & 1 & 5 & 3 \end{pmatrix},$$

in S_8.
Solution:
For finding the orbit containing 1, we apply σ repeatedly, obtaining symbolically

$$1 \xrightarrow{\sigma} 2 \xrightarrow{\sigma} 8 \xrightarrow{\sigma} 3 \xrightarrow{\sigma} 6 \xrightarrow{\sigma} 1 \xrightarrow{\sigma} 2 \xrightarrow{\sigma} 8 \xrightarrow{\sigma} 3 \xrightarrow{\sigma} 6 \xrightarrow{\sigma} 1 \xrightarrow{\sigma} 2 \xrightarrow{\sigma} 8 \ldots.$$

Since σ^{-1} would simply reverse the directions of the arrows in this chain, we see that the orbit containing 1 is $\{1,2,8,3,6\}$. We now choose an integer from 1 to 8 not in $\{1,2,8,3,6\}$, say 4, and similarly find that the orbit containing 4 is $\{4,7,5\}$. Since these two orbits include all integers from 1 to 8, we see that the complete list of orbits of σ is

$$\{1,2,8,3,6\}, \ \{4,7,5\}.$$

Definition 2.5. If the group G acts on the set X, then the subgroup

$$G_x = \{g \in G \mid g.x = x\}$$

is called the **isotropy subgroup** of x or stabilizer of x.

Now, we shall find a basic relationship between the order of an $orb_G(x)$ and the order of the isotropy subgroup G_x.

Let G is a group of permutations of a set X, and let x be any element of X. The set T of pairs (g, y) such that $g(x) = y$ can be shown by means of a table as in Section 3.2[2].

	.	.	.	y	...	Row total
.	$\surd$		$\surd$		$\surd$	.
.						.
.	$\surd$	$\surd$		$\surd$		.
g					$\surd$	$r_g(T)$
.	$\surd$ means that			$\surd$		.
.	$(g,\, y)$ is in T		$\surd$			.
.		$\surd$			$\surd$	.
Column total	.	.	.	$c_y(T)$	...	

The two ways of counting T, using the row totals $r_g(T)$ and the column totals $c_y(T)$, provide the basis for the proof of Lemma 2.3.

Lemma 2.3. *Let X be a set and G a group of permutations acting on X. Then G_x is a subgroup of G, for each $x \in X$. If in addition G is finite, then*

$$|orb_G(x)| \times |G_x| = |G|.$$

Proof. Let T denote the set of pairs described in the table above, that is

$$T = \{(g,y) \mid g(x) = y\}.$$

As g is a permutation there is just one y such that $g(x) = y$, for each g. In other words, each row total $r_g(T)$ is equal to 1.

The column total $c_y(T)$ is the number of g such that $g(x) = y$, that is $|G(x \to y)|$. Then if y is in the $orb_G(x)$ we have

$$c_y(T) = |G(x \to y)| = |G_x|.$$

On the other hand, if y is not in $orb_G(x)$ there are no permutations in G which take x to y, and so $c_y(T) = 0$.

From the two methods for counting T, we have

$$\sum_{y \in X} c_y(T) = \sum_{g \in G} r_g(T).$$

Where as on the left-hand side there are $|orb_G(x)|$ terms equal to $|G_x|$ and the remaining are zero, but on the right-hand side there are $|G|$ terms equal to 1. Hence the result. □

2.3 Cycle Index

Let G be a permutation group of degree n. Each permutation g in G has a unique decomposition into disjoint cycles, say $c_1c_2c_3.......$ Let the length of a cycle c be denoted by $|c|$. Now let $j_k(g)$ be the number of cycles of g of length k, where

$$0 \leq j_k(g) \leq \lfloor n/k \rfloor \;\; and \;\; \sum_{k=1}^{n} k j_k(g) = n.$$

We associate to g the monomial

$$\prod_{c \in g} a_{|c|} = \prod_{k=1}^{n} a_k^{j_k(g)}$$

in the variables $a_1, a_2, \ldots\ldots., a_n$.

Definition 2.6. The **cycle index** of a permutation group G is the average of

$$a_1^{j_1(g)} a_2^{j_2(g)} a_3^{j_3(g)} \ldots\ldots$$

over all permutations g of the group, where $j_k(g)$ is the number of cycles of length k in the disjoint cycle decomposition of g.

The cycle index $Z(G)$ of G is given by

$$Z(G) = \frac{1}{|G|} \sum_{g \in G} \prod_{k=1}^{n} a_k^{j_k(g)}.$$

2.4 Generating function

Definition 2.7. The generating function for the infinite sequence $(f_0, f_1, f_2, f_3, \ldots)$ is the formal power series

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \ldots.$$

A **generating function** is a "formal" power series in the sense that we usually regard x as a placeholder instead of a number. Only in rare cases will we let x be a real number and actually evaluate a generating function, so we can largely forget about questions of convergence. Not all generating functions are ordinary, but those are the only kind we will consider here.

Example 2.7. Perhaps the best example of a generating function arises from the binomial theorem. The formula

$$(1+x)^k = \binom{k}{0} + \binom{k}{1} x + \binom{k}{2} x^2 + \ldots + \binom{k}{n} x^n + \ldots,$$

can be regarded as saying that the generating function for the sequence defined by

$$f_n = \binom{k}{n} \quad \text{where} \quad f_n = 0 \;\text{ for } n > k,$$

for any given integer k, is

$$F(x) = (1+x)^k.$$

3 Polya's Enumeration Theorem

In this section, we shall prove Polya's enumeration theorem and Burnside's lemma.

Suppose G is a group of permutations of a set X, and let $\hat{G}$ be the induced group of permutations of the set Ψ of colorings of X. Now each permutation g in G induces a permutation $\hat{g}$ of Ψ in the following way. Given a coloring ω, we define $\hat{g}(\omega)$ to be the coloring in which the color assigned to x is the color ω assigns to $g(x)$; that is,

$$(\hat{g}(\omega))(x) = \omega(g(x)).$$

We require the generating function $K_E(c_1,c_2,...,c_k)$, where E is a set of colorings containing one representative of each orbit of $\hat{G}$ on Ψ. The coefficient of $c_1^s c_2^t....$ in K_E will be the number of distinguishable colorings in which color c_1 is used s times, color c_2 is used t times, and so on.
Polya's theorem state that K_E is obtained from the cycle index $Z_G(a_1,a_2,...,a_n)$ by substituting

$$c_1^i + c_2^i + ... + c_k^i$$

for a_i $(1 \leq i \leq n)$. Before going to the proof, let us see how this works in the simple case of the red-and-white colorings of the corners of a square.

Example 3.1. As an example, let us consider the number of ways of assigning one of the colors red or white to each corner of a square. Since there are two colors and four corners there are basically $2^4 = 16$ possibilities. But when we take account of the symmetry of the square we see that some of the possibilities are essentially the same. For example, the first coloring as in figure below is the same as the second one after rotation through 180^0.

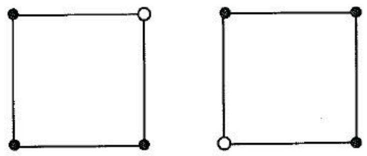

Figure 3.1: Two indistinguishable colorings

From above, we regard two colorings as being indistinguishable if one is transformed into the other by a symmetry of the square. It is easy to find the distinguishable colorings (in this example) by trial and error: there are just six of them, as shown in the figure below.

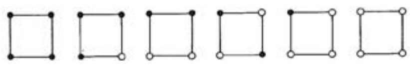

Figure 3.2: The six distinguishable colorings

Let G be a group of permutations of a set X, where frequently we take X to be the set $\{1,2,...,n\}$. Each element g in G can be written in cycle notation with j_i cycles of length i $(1 \leq i \leq n)$, and we recall that the type of g is the corresponding partition

$$[1^{j_1} 2^{j_2} ... n^{j_n}]$$

of n. Of course, we have $j_1 + 2j_2 + ... + nj_n = n$. We shall associate with g an expression

$$Z_g(a_1,a_2,...,a_n) = a_1^{j_1} a_2^{j_2} ... a_n^{j_n},$$

where the a_i $(1 \leq i \leq n)$ are, for the moment, simply formal symbols like the x in a polynomial. For example, if G is the group of symmetries of a square, regarded as permutations of the corners $1,2,3,4$ then the expression Z_g are given in Table 3.1. Note that although the 1-cycles are conventionally omitted in the notation for g it is important to include them in Z_g.

g	j_1	j_2	j_3	j_4	Z_g
id	4	-	-	-	a_1^4
(1234)	-	-	-	1	a_4
(13)(24)	-	2	-	-	a_2^2
(1432)	-	-	-	1	a_4
(12)(34)	-	2	-	-	a_2^2
(14)(23)	-	2	-	-	a_2^2
(13)	2	1	-	-	$a_1^2 a_2$
(24)	2	1	-	-	$a_1^2 a_2$

Table 3.1:

The formal sum of the Z_g, taken over all g in G, is a 'polynomial' in $a_1, a_2, ..., a_n$. Dividing by $|G|$ we obtain the cycle index of the group of permutations:

$$Z_g(a_1, a_2, ..., a_n) = \frac{1}{|G|} \sum_{g \in G} Z_g(a_1, a_2, ..., a_n).$$

For example, the cycle index of the group of the square is, as considered above, is

$$\frac{1}{8}(a_1^4 + 2a_1^2 a_2 + 3a_2^2 + 2a_4),$$

and we have to substitute

$$a_1 = r + w, \quad a_2 = r^2 + w^2, \quad a_3 = r^3 + w^3, \quad a_4 = r^4 + w^4.$$

We get

$$\begin{aligned} K_E(r, w) &= \frac{1}{8}[(r+w)^4 + 2(r+w)^2(r^2+w^2) + 3(r^2+w^2)^2 + 2(r^4+w^4)] \\ &= r^4 + r^3 w + 2r^2 w^2 + r w^3 + w^4. \quad \text{(By using Mathematica)} \end{aligned}$$

Definition 3.1. Suppose we have a group G of permutations of an n-set X, and every element of X can be assigned one of r colors. Let we denote the set of colors by E, then a **coloring** is simply a function ω from X to E. There are r^n colorings in all, and we shall denote the set of them by Ψ.

Now each permutation g in G induces a permutation $\hat{g}$ of Ψ in the following way. Let a coloring ω, we define $\hat{g}(\omega)$ to be the coloring in which the color assigned to x is the color ω assigns to $g(x)$; so,

$$(\hat{g}(\omega))(x) = \omega(g(x)).$$

The definition is described in Fig. 3.3, where g is the clockwise rotation through 90^0, and ω is the coloring on the right-hand side.

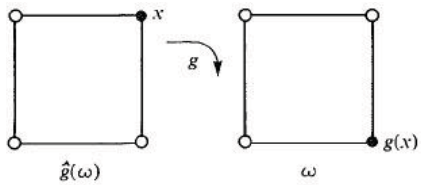

Figure 3.3: Describing the definition of $\hat{g}(\omega)$

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Theorem 3.1. *The number of orbits of G on X is*

$$\frac{1}{|G|}\sum_{g\in G}|F(g)| \quad where, \;\; F(g)=\{x\in X,\ g\in G \mid g(x)=x\}.$$

Its weighted form is

$$\sum_{x\in D} I(x)=\frac{1}{|G|}\sum_{g\in G}\sum_{x\in F(g)} I(x).$$

Proof. Let G be a group of permutations of X, and let $I(x)$ be an expression which is constant on each orbit of G, so that

$$I(g(x)) = I(x) \text{ for all } g\in G,\ x\in X.$$

Let D be a set of representatives, one from each orbit, and let $E=\{(g,x)\mid g(x)=x\}$, as in the proof of Theorem 14.4[2]. By evaluating the sum

$$\sum_{(g,x)\in E} I(x)$$

in two different ways (as described in Lemma 2.3), show that

$$\sum_{x\in D} I(x)=\frac{1}{|G|}\sum_{g\in G}\sum_{x\in F(g)} I(x).$$

This is the required 'weighted' version.

□

Suppose that the set X is to be colored, and that the set of available colors is $U=\{c_1,c_2,\ldots,c_n\}$. Associated with each coloring $\omega: X\to U$ is a formal expression, the **indicator** of ω, defined by

$$ind(\omega)=c_1^{h_{c_1}}c_2^{h_{c_2}}\ldots c_n^{h_{c_n}},$$

where $h_{c_1},h_{c_2},\ldots,h_{c_n}$ are the numbers of members of X which receive colors $c_1,c_2,\ldots,c_n$ respectively. Here, $h_{c_1}+h_{c_2}+\ldots+h_{c_n}=h$, and $h=|X|$.

Let any subset B of the set Ψ of all colorings, we define the generating function K_B to be the formal sum

$$K_B(c_1,c_2,\ldots,c_n)=\sum_{\omega\in B} ind(\omega).$$

Clearly, when the terms of K_B are collected in the normal way, the coefficient of the term $c_1^s c_2^t\ldots$ is just the number of colorings in B in which color c_1 is used s times, color c_2 is used t times, and so on.

Now, we are able to prove Polya's Enumeration Theorem (PET).

Theorem 3.2. *Let $Z_G(a_1,a_2,...,a_n)$ be the cycle index for a group G of permutations of X. The generating function $K_E(c_1,c_2,...,c_n)$ for the numbers of inequivalent colorings of X, when the colors available are $c_1,c_2,...,c_n$, is given by*

$$K_E(c_1,c_2,...,c_n) = Z_G(\tau_1,\tau_2,...,\tau_n),$$

where

$$\tau_i = c_1^i + c_2^i + ... + c_n^i \quad (1 \leq i \leq n).$$

Proof. We shall begin by finding an alternative formula for

$$K_E(c_1,c_2,...,c_n) = \sum_{\omega \in E} ind(\omega),$$

where E is a set of colorings containing one representative of each orbit of $\hat{G}$ on Ψ. We will do this by using 'weighted' form of the Theorem 3.1.
Applying this result to the action of $\hat{G}$ on Ψ, we get

$$\sum_{\omega \in D} ind(\omega) = \frac{1}{|\hat{G}|} \sum_{\hat{g} \in \hat{G}} \left[\sum_{\omega \in F(\hat{g})} ind(\omega) \right].$$

Now the sum in the bracket is just $K_{F(\hat{g})}$, by definition. Further more, a coloring ω is in $F(\hat{g})$ if and only if it is constant on each cycle of g. Hence the explicit form of $K_{F(\hat{g})}$ is given by

$$\begin{aligned} K_{F(\hat{g})}(c_1,c_2,...,c_n) &= (c_1^{m_1} + ... + c_n^{m_1}) \cdot ... \cdot (c_1^{m_k} + ... + c_n^{m_k}) \\ &= \tau_{m_1} ... \tau_{m_k}, \end{aligned}$$

where $m_1,m_2,...,m_k$ are the lengths of the cycles of g. In other words, if g has j_i cycles of length i $(1 \leq i \leq n)$ then

$$\begin{aligned} K_{F(\hat{g})}(c_1,c_2,...,c_n) &= \tau_1^{j_1} \tau_2^{j_2} ... \tau_n^{j_n} \\ &= Z_g(\tau_1,\tau_2,...,\tau_n). \end{aligned}$$

Since the representation $g \to \hat{g}$ is a bijection, we have $|G| = |\hat{G}|$, and substituting for $K_{F(\hat{g})}$above we get

$$K_E(c_1,c_2,...,c_n) = Z_g(\tau_1,\tau_2,...,\tau_n),$$

as required. □

3.1 Burnside's lemma

Theorem 3.3. *Let G be a finite group that acts on the finite set X. Let r denote the number of orbits in X under the action of G. Then*

$$r = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

Proof. Suppose the set $M = \{(g,x) \in G \times X \mid g.x = x\}$ contains m elements. The idea is to count the elements of M in two different ways, and thereby obtain two different expressions, both equal to m. Combining these expressions will yield Burnside's Lemma. Now M contains all pairs (g,x) such that $g.x = x$. We recall that the set X_g for each fixed

$g \in G$ contains all $x \in X$ such that $g.x = x$. Thus, for each fixed $g \in G$, there must be $|X_g|$ elements x that fulfills $g.x = x$. Therefore

$$m = \sum_{g \in G} |X_g|.$$

The isotropy subgroup G_x contains, on the other hand, those elements $g \in G$ fulfilling $g.x = x$, for each fixed $x \in X$. So for each fixed x, there are $|G_x|$ elements g such that $g.x = x$. This yields

$$m = \sum_{x \in X} |G_x|.$$

According to the Lemma 2.3, $|orb_G(x)| = |G|/|G_x|$. Thus

$$\sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|orb_G(x)|} = |G| \sum_{x \in X} \frac{1}{|orb_G(x)|}.$$

Let $B_1, B_2, ..., B_r$ denote all orbits in X. For each $x \in X$ we thus have $orb_G(x) = B_i$ for some i. Since $X = B_1 \cup B_2 \cup ... \cup B_r$ and $B_i \cap B_j = \phi$ on $i \neq j$ (the orbits are equivalence classes), we find that

$$\sum_{x \in X} \frac{1}{|orb_G(x)|} = \sum_{x \in B_1} \frac{1}{|B_1|} + \sum_{x \in B_2} \frac{1}{|B_2|} + ... + \sum_{x \in B_r} \frac{1}{|B_r|}.$$

But for each i,

$$\sum_{x \in B_i} \frac{1}{|B_i|} = |B_i|.\frac{1}{|B_i|} = 1,$$

and therefore

$$m = |G| \sum_{x \in X} \frac{1}{|orb_G(x)|} = |G|(1 + 1 + ... + 1) = |G|.r.$$

In the beginning of proof we also found that $m = \sum_{g \in G} |X_g|$. If we combine these expressions for m, we obtain an equation which we solve for r, in order to obtain Burnside's Formula.
Which is

$$r = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

□

Remark:
The number of colorings by Burnside's lemma is the same as the number of colorings by Polya's enumeration theorem, if we put 1 instead of a color in generating function of PET. We will show this in the next examples.

4 Colorings of the square and the regular pentagon

In this section, we use Burnside's lemma and PET to investigate the number of colorings of the square and the regular pentagon.

4.1 Colorings of corners of a square

Example 4.1. In how many ways, can we color the corners of the square with two colors.

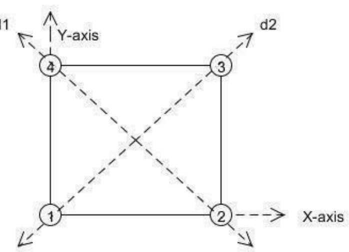

Figure 4.1: Square

Here the permutation group is D_4.
We have two colors. First we find all possible permutations under the action of rotation and reflection, and set of all permutations are isomorphic to D_4.
So,

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)(2)(3)(4).$$

Possible ways (Number of coloring of square's corners) under action ε are

$$|X_\varepsilon| = 2^4 = 16.$$

We have

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1\,2\,3\,4) \quad (90^0 \text{ counter clockwise})$$
$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1\,3)(2\,4) \quad (180^0 \text{ counter clockwise})$$
$$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1\,4\,3\,2) \quad (90^0 \text{ clockwise}).$$

So, possible ways (Number of coloring of square's corners) under action ρ_1, ρ_2, ρ_3 are

$$|X_{\rho_1}| = |X_{\rho_3}| = 2^1 = 2$$
$$|X_{\rho_2}| = 2^2 = 4.$$

By reflections of square along axis, we have

$$\sigma_x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (1\,4)(2\,3) \quad (\text{rotation along x-axis})$$
$$\sigma_y = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1\,2)(3\,4) \quad (\text{rotation along y-axis}).$$

So, possible ways (Number of coloring of square's corners) under action σ_x and σ_y are

$$|\sigma_x| = |\sigma_y| = 2^2 = 4.$$

By reflections of square along digonals, we have

$$\sigma_{d_1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = (1\,3)(2)(4) \quad \text{(rotation along } d_1 \text{ diagonal)}$$

$$\sigma_{d_2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (1)(3)(2\,4) \quad \text{(rotation along } d_2 \text{ diagonal).}$$

So, possible ways (Number of coloring of square's corners) under action σ_{d_1} and σ_{d_2} are

$$|\sigma_{d_1}| = |\sigma_{d_2}| = 2^3 = 8.$$

Then by Theorem 3.3, we have

$$\begin{aligned} \text{required number of colorings} &= \frac{1}{8}(16+2\cdot 2+4+2\cdot 4+2\cdot 8) \\ &= \frac{1}{8}(16+4+4+8+16) \\ &= \frac{1}{8}(24+24) \\ &= \frac{1}{8}(48) \\ &= 6 \end{aligned}$$

Now, we are going to use PET for two colors in a square.
From above, we have the following cycle structures of permutations

(1)(2)(3)(4)
(1 2 3 4)
(1 3)(2 4)
(1 4 3 2)
(1 4)(2 3)
(1 2)(3 4)
(1 3)(2)(4)
(1)(3)(2 4)

Table 4.1: Cycle structure of D_4

so the cycle index of D_4 is

$$Z(D_4) = \frac{1}{8}(1\cdot a_1^4 + 3a_2^2 + 2a_1^2 a_2 + 2a_4). \tag{4.1}$$

Generating function coloring one corner is

$$F(X,Y) = 1\cdot X + 1\cdot Y.$$

We can use color X or Y to color a corner in the square.
By Polya's Enumeration Theorem (PET), we have

$$Z(D_4)(X,Y) = \frac{1}{8}[((X+Y)^4 + 3(X^2+Y^2)^2 + 2(X+Y)^2(X^2+Y^2) + 2(X^4+Y^4)].$$

Generating function for the colorings of the square. By using Mathematica we have

$$Z(D_4)(X,Y) = X^4 + Y^4 + X^3Y + XY^3 + 2X^2Y^2$$
$$Z(D_4)(1,1) = 1+1+1+1+2 = 6. \quad \text{(replacing colors by 1)}$$

So, total number of colorings = 6.
Hence, the possible colorings are:
1- All corners have color X
1- All corners have color Y
1- Three corners have color X and one has Y
1- One corner has color X and three have Y
2- Two corners have color X and two have Y.

Example 4.2. Now we solve Example 4.1 for 3 colors.
We have 2 colors in Example 4.1 but here we have 3 colors, so replacing 2 by 3 in the base of number of colorings, we have

$$
\begin{aligned}
|X_\varepsilon| &= 3^4 = 81 \\
|X_{\rho_1}| = |X_{\rho_3}| &= 3^1 = 3 \\
|X_{\rho_2}| &= 3^2 = 9 \\
|\sigma_x| = |\sigma_y| &= 3^2 = 9 \\
|\sigma_{d_1}| = |\sigma_{d_2}| &= 3^3 = 27.
\end{aligned}
$$

Then by Theorem 3.3, we have

$$
\begin{aligned}
\text{required number of colorings} &= \frac{1}{8}(81 + 2\cdot 3 + 9 + 2\cdot 9 + 2\cdot 27) \\
&= \frac{1}{8}(81 + 6 + 9 + 18 + 54) \\
&= \frac{1}{8}(168) \\
&= 21.
\end{aligned}
$$

From equation 4.1 cycle index of D_4 is

$$Z(D_4) = \frac{1}{8}(1\cdot a_1^4 + 3a_2^2 + 2a_1^2 a_2 + 2a_4)$$

Generating function coloring one corner is

$$F(X,Y,Z) = 1\cdot X + 1\cdot Y + 1\cdot Z$$

This means that, we can use color X, Y or Z to color a corner of the square.
By Polya's Enumeration Theorem (PET), we have

$$Z(D_4)(X,Y,Z) = \frac{1}{8}[(X+Y+Z)^4 + 3(X^2+Y^2+Z^2)^2 + 2(X+Y+Z)^2(X^2+Y^2+Z^2) + 2(X^4+Y^4+Z^4)].$$

Generating function for the colorings of the square. By using Mathematica we have

$$
\begin{aligned}
Z(D_4)(X,Y,Z) =& X^4 + X^3Y + 2X^2Y^2 + XY^3 + Y^4 + X^3Z + 2X^2YZ + 2XY^2Z + Y^3Z + \\
& 2X^2Z^2 + 2XYZ^2 + 2Y^2Z^2 + XZ^3 + YZ^3 + Z^4.
\end{aligned}
$$

$$Z(D_4)(1,1,1) = 1+1+2+1+1+1+2+2+1+2+2+2+1+1+1 = 21.$$

So total number of colorings $= 21$.
Hence from above we can say that, when we use two colors at a time, the possible colorings are 12.

Example 4.3. Now, we solve Example 4.1 for 4 different colors.
We have 2 colors in Example 4.1 but here we have 4 colors, so replacing 2 by 4 in the base of number of colorings, we have

$$
\begin{aligned}
|X_\varepsilon| &= 4^4 = 256 \\
|X_{\rho_1}| &= |X_{\rho_3}| = 4^1 = 4 \\
|X_{\rho_2}| &= 4^2 = 16 \\
|\sigma_x| &= |\sigma_y| = 4^2 = 16 \\
|\sigma_{d_1}| &= |\sigma_{d_2}| = 4^3 = 64.
\end{aligned}
$$

Then by Theorem 3.3, we have

$$
\begin{aligned}
\text{Required number of colorings} &= \frac{1}{8}(256 + 2\cdot 4 + 16 + 2\cdot 16 + 2\cdot 64) \\
&= \frac{1}{8}(440) \\
&= 55.
\end{aligned}
$$

From equation 4.1 cycle index of D_4 is

$$Z(D_4) = \frac{1}{8}(1\cdot a_1^4 + 3a_2^2 + 2a_1^2\cdot a_2 + 2a_4)$$

Generating function coloring one corner is

$$F(X,Y,Z,U) = 1\cdot X + 1\cdot Y + 1\cdot Z + 1\cdot U$$

we can use color X, Y, Z or U to color a corner of the square.
By Polya's Enumeration Theorem(PET), we have

$$
\begin{aligned}
Z(D_4)(X,Y,Z,U) =& \frac{1}{8}[(X+Y+Z+U)^4 + 3(X^2+Y^2+Z^2+U^2)^2 + \\
& 2(X+Y+Z+U)^2(X^2+Y^2+Z^2+U^2) + 2(X^4+Y^4+Z^4+U^4)].
\end{aligned}
$$

Generating function for the colorings of the square. By using Mathematica we have

$$
\begin{aligned}
Z(D_4)(X,Y,Z,U) =& U^4 + U^3X + 2U^2X^2 + UX^3 + X^4 + U^3Y + 2U^2XY + 2UX^2Y + \\
& X^3Y + 2U^2Y^2 + 2UXY^2 + 2X^2Y^2 + UY^3 + XY^3 + Y^4 + \\
& U^3Z + 2U^2XZ + 2UX^2Z + X^3Z + 2U^2YZ + 3UXYZ + 2X^2YZ + \\
& 2UY^2Z + 2XY^2Z + Y^3Z + 2U^2Z^2 + 2UXZ^2 + 2X^2Z^2 + \\
& 2UYZ^2 + 2XYZ^2 + 2Y^2Z^2 + UZ^3 + XZ^3 + YZ^3 + Z^4.
\end{aligned}
$$

$$Z(D_4)(1,1,1,1) = 55.$$

So total number of colorings $= 55$.
Hence from above we can say that, when we use three colors at a time, the possible colorings are 24.

4.2 Colorings of sides of a pentagon

Example 4.4. In how many ways, we can color sides of a pentagon with three colors.
Solution
We let the dihedral group D_5 act on the set of all colorings.

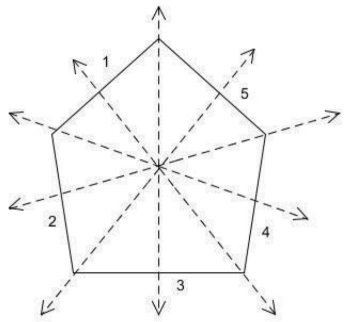

Figure 4.2: pentagon

Then
ρ_k =rotation clockwise by $k.72^0$, $k = 0,1,2,3,4$
μ_k =reflection in the line of symmetry that passes through the mid point of side, $k = 1,2,3,4,5$.
So

$$|X_{\rho_0}| = 3^5 \text{ (no rotation)}$$
$$|X_{\rho_k}| = 3 \text{ (all sides must have same color)}, \ k = 1,2,3,4$$
$$|X_{\mu_k}| = 3^3.$$

By Theorem 3.3, we have

$$\begin{aligned}\text{required number of colorings} &= \frac{1}{10}(3^5 + 4\cdot 3 + 5\cdot 3^3)\\ &= \frac{1}{10}(243 + 12 + 135)\\ &= \frac{1}{10}(390)\\ &= 39.\end{aligned}$$

From above, we have the following cycle structures of permutations.

$\rho_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = (1)(2)(3)(4)(5).$
$\rho_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix} = (1\ 5\ 4\ 3\ 2).$
$\rho_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix} = (1\ 4\ 2\ 5\ 3).$
$\rho_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix} = (1\ 3\ 5\ 2\ 4).$
$\rho_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} = (1\ 2\ 3\ 4\ 5).$
$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} = (1\ 5)(2\ 4)(3).$
$\mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix} = (1)(2\ 5)(3\ 4).$
$\mu_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} = (1\ 3)(2)(4\ 5).$
$\mu_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix} = (1\ 2)(3\ 5)(4).$
$\mu_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix} = (1\ 4)(2\ 3)(5).$

Table 4.2: Cycle structure of D_5

so the cycle index of D_5 is

$$Z(D_5) = \frac{1}{10}(1 \cdot a_1^5 + 4a_5 + 5a_2^2 a_1).$$

Generating function coloring one cornor is

$$F(X,Y,Z) = 1 \cdot X + 1 \cdot Y + 1 \cdot Z\,.$$

We can use color x, y or z to color a corner of the pentagon.

By Polya's Enumeration Theorem (PET), we have

$$Z(D_5)(X,Y,Z) = \frac{1}{10}[(X+Y+Z)^5 + 4(X^5+Y^5+Z^5) + 5(X^2+Y^2+Z^2)^2(X+Y+Z)].$$

By using Mathematica, we have

$$\begin{aligned} Z(D_5)(X,Y,Z) =& X^5 + X^4Y + 2X^3Y^2 + 2X^2Y^3 + XY^4 + Y^5 + X^4Z + 2X^3YZ + 4X^2Y^2Z + \\ & 2XY^3Z + Y^4Z + 2X^3Z^2 + 4X^2YZ^2 + 4XY^2Z^2 + 2Y^3Z^2 + \\ & 2X^2Z^3 + 2XYZ^3 + 2Y^2Z^3 + XZ^4 + YZ^4 + Z^5. \\ Z(D_5)(1,1,1) =& 1+1+2+2+1+1+1+2+4+2+1+2+4+4+ \\ & 2+2+2+2+1+1+1. \\ Z(D_5)(1,1,1) =& 39. \end{aligned}$$

5 Colorings of regular n-gons

Now, we investigate the cycle structure of the rotations in $D_6, D_7, D_8, D_9, D_{10}$. Then we will find the general expression for D_n.

5.1 Cycle index for the group of rotations of n-gons

Definition 5.1. The Euler totient $\phi(n)$ of a positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n (i.e having no common positive factors other than 1). In particular $\phi(1) = 1$ since 1 is coprime to itself (1 being the only natural number with this property).
i.e
$\phi(n)$ =Euler's totient function.
$\phi(n)$ =The number of integers in $\{1,2,3,...,n\}$ that are relatively prime to n.
or
$\phi(n) = |\{x \in \{1,2,3,...,n\} : gcd(x,n) = 1\}|$.
For example
$\phi(5) = |\{x \in \{1,2,3,4,5\} : gcd(x,5) = 1\}| = |\{1,2,3,4\}| = 4$.
$\phi(9) = |\{x \in \{1,2,3,4,5,6,7,8,9\} : gcd(x,9) = 1\}| = |\{1,2,4,5,7,8\}| = 6$.

Rotations for D_6

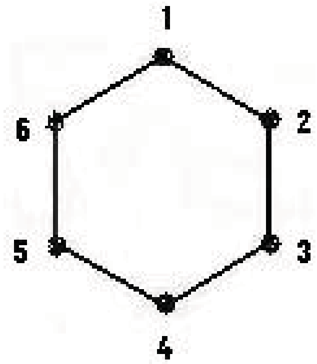

Figure 5.1: Hexagon

Cycle structure of permutations of rotations for D_6 is

ε	$= (1)(2)(3)(4)(5)(6)$
ρ_1	$= (1\ 2\ 3\ 4\ 5\ 6)$
ρ_2	$= (1\ 3\ 5)(2\ 4\ 6)$
ρ_3	$= (1\ 4)(2\ 5)(3\ 6)$
ρ_4	$= (1\ 5\ 3)(2\ 6\ 4)$
ρ_5	$= (1\ 6\ 5\ 4\ 3\ 2)$

Table 5.1: Cycle structure for D_6

Comparing, we have
Cycle index= $1 \cdot a_1^6 + 1 \cdot a_2^3 + 2 \cdot a_3^2 + 2 \cdot a_6 = \sum_{d/6} a_d^{6/d}$.
Cycle index= $\phi(1)a_1^6 + \phi(2)a_2^3 + \phi(3)a_3^2 + \phi(6)a_6$.

Since
$\phi(1) = |\{x \in \{1\} : gcd(x,1) = 1\}| = |\{1\}| = 1.$
$\phi(2) = |\{x \in \{1,2\} : gcd(x,2) = 1\}| = |\{1\}| = 1.$
$\phi(3) = |\{x \in \{1,2,3\} : gcd(x,3) = 1\}| = |\{1,2\}| = 2.$
$\phi(6) = |\{x \in \{1,2,3,4,5,6\} : gcd(x,6) = 1\}| = |\{1,5\}| = 2.$
Hence

$$Z_{c_6} = 1 \cdot a_1^6 + 1 \cdot a_2^3 + 2 \cdot a_3^2 + 2 \cdot a_6.$$

Rotations for D_7

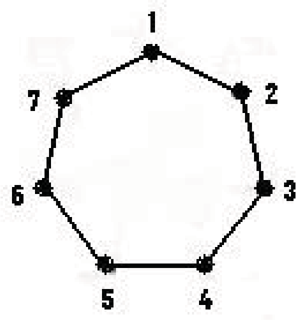

Figure 5.2: Heptagon

Cycle structure of permutations of rotations for D_7 is

$\varepsilon = (1)(2)(3)(4)(5)(6)(7).$
$\rho_1 = (1\ 2\ 3\ 4\ 5\ 6\ 7).$
$\rho_2 = (1\ 3\ 5\ 7\ 2\ 4\ 6).$
$\rho_3 = (1\ 4\ 7\ 3\ 6\ 2\ 5).$
$\rho_4 = (1\ 5\ 2\ 6\ 3\ 7\ 4).$
$\rho_5 = (1\ 6\ 4\ 2\ 7\ 5\ 3).$
$\rho_6 = (1\ 7\ 6\ 5\ 4\ 3\ 2).$

Table 5.2: Cycle structure for D_7

Comparing, we have
Cycle index= $1 \cdot a_1^7 + 6 \cdot a_7^1 = \sum_{d/7} a_d^{7/d}.$
Cycle index= $\phi(1)a_1^7 + \phi(7)a_7.$
Since
$\phi(1) = |\{x \in \{1\} : gcd(x,1) = 1\}| = |\{1\}| = 1.$
$\phi(7) = |\{x \in \{1,2,3,4,5,6,7\} : gcd(x,7) = 1\}| = |\{1,2,3,4,5,6\}| = 6.$
Hence

$$Z_{c_7} = 1 \cdot a_1^7 + 6 \cdot a_7.$$

Rotations for D_8
Cycle structure of permutations of rotations for D_8 is

$\varepsilon \ = (1)(2)(3)(4)(5)(6)(7)(8).$
$\rho_1 = (1\,2\,3\,4\,5\,6\,7\,8).$
$\rho_2 = (1\,3\,5\,7)(2\,4\,6\,8).$
$\rho_3 = (1\,4\,7\,2\,5\,8\,3\,6).$
$\rho_4 = (1\,5)(2\,6)(3\,7)(4\,8).$
$\rho_5 = (1\,6\,3\,8\,5\,2\,7\,4).$
$\rho_6 = (1\,7\,5\,3)(2\,8\,6\,4).$
$\rho_7 = (1\,8\,7\,6\,5\,4\,3\,2).$

Table 5.3: Cycle structure for D_8

Comparing, we have
Cycle index= $1 \cdot a_1^8 + 1 \cdot a_2^4 + 2 \cdot a_4^2 + 4 \cdot a_8 = \sum_{d/8} a_d^{8/d}$.
Cycle index= $\phi(1)a_1^8 + \phi(2)a_2^4 + \phi(4)a_4^2 + \phi(8)a_8$.
Since
$\phi(1) = |\{x \in \{1\} : gcd(x,1) = 1\}| = |\{1\}| = 1.$
$\phi(2) = |\{x \in \{1,2\} : gcd(x,2) = 1\}| = |\{1\}| = 1.$
$\phi(4) = |\{x \in \{1,2,3,4\} : gcd(x,4) = 1\}| = |\{1,3\}| = 2.$
$\phi(8) = |\{x \in \{1,2,3,4,5,6,7,8\} : gcd(x,8) = 1\}| = |\{1,3,5,7\}| = 4.$
Hence
$$Z_{c_8} = 1 \cdot a_1^8 + 1 \cdot a_2^4 + 2 \cdot a_4^2 + 4 \cdot a_8 \ .$$

Rotations for D_9
Cycle structure of permutations of rotations for D_9 is

$\varepsilon \ = (1)(2)(3)(4)(5)(6)(7)(8)(9).$
$\rho_1 = (1\,2\,3\,4\,5\,6\,7\,8\,9).$
$\rho_2 = (1\,3\,5\,7\,9\,2\,4\,6\,8).$
$\rho_3 = (1\,4\,7)(2\,5\,8)(3\,6\,9).$
$\rho_4 = (1\,5\,9\,4\,8\,3\,7\,2\,6).$
$\rho_5 = (1\,6\,2\,7\,3\,8\,4\,9\,5).$
$\rho_6 = (1\,7\,4)(2\,8\,5)(3\,9\,6).$
$\rho_7 = (1\,8\,6\,4\,2\,9\,7\,5\,3).$
$\rho_8 = (1\,9\,8\,7\,6\,5\,4\,3\,2).$

Table 5.4: Cycle structure for D_9

Comparing, we have
Cycle index= $1 \cdot a_1^9 + 2 \cdot a_3^3 + 6 \cdot a_9 = \sum_{d/9} a_d^{9/d}$.
Cycle index= $\phi(1)a_1^9 + \phi(3)a_3^3 + \phi(9)a_9$.
Since
$\phi(1) = |\{x \in \{1\} : gcd(x,1) = 1\}| = |\{1\}| = 1.$
$\phi(3) = |\{x \in \{1,2,3\} : gcd(x,3) = 1\}| = |\{1,2\}| = 2.$
$\phi(9) = |\{x \in \{1,2,3,4,5,6,7,8,9\} : gcd(x,9) = 1\}| = |\{1,2,4,5,7,8\}| = 6.$
Hence
$$Z_{c_9} = 1 \cdot a_1^9 + 2 \cdot a_3^3 + 6 \cdot a_9 \ .$$

Rotations for D_{10}
Cycle structure of permutations of rotations for D_{10} is

ε	$= (1)(2)(3)(4)(5)(6)(7)(8)(9)(10).$
ρ_1	$= (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10).$
ρ_2	$= (1\ 3\ 5\ 7\ 9)(2\ 4\ 6\ 8\ 10).$
ρ_3	$= (1\ 4\ 7\ 10\ 3\ 6\ 9\ 2\ 5\ 8).$
ρ_4	$= (1\ 5\ 9\ 3\ 7)(2\ 6\ 10\ 4\ 8).$
ρ_5	$= (1\ 6)(2\ 7)(3\ 8)(4\ 9)(5\ 10).$
ρ_6	$= (1\ 7\ 3\ 9\ 5)(2\ 8\ 4\ 10\ 6).$
ρ_7	$= (1\ 8\ 5\ 2\ 9\ 6\ 3\ 10\ 7\ 4).$
ρ_8	$= (1\ 9\ 7\ 5\ 3)(2\ 10\ 8\ 6\ 4).$
ρ_9	$= (1\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2).$

Table 5.5: Cycle structure for D_{10}

Comparing, we have
Cycle index= $1\cdot a_1^{10}+1\cdot a_2^5+4\cdot a_5^2+4\cdot a_{10}=\sum_{d/10} a_d^{10/d}$.
Cycle index= $\phi(1)a_1^{10}+\phi(2)a_2^5+\phi(5)a_5^2+\phi(10)a_{10}$. Since
$\phi(1)=|\{x\in\{1\}:gcd(x,1)=1\}|=|\{1\}|=1.$
$\phi(2)=|\{x\in\{1,2\}:gcd(x,2)=1\}|=|\{1\}|=1.$
$\phi(5)=|\{x\in\{1,2,3,4,5\}:gcd(x,5)=1\}|=|\{1,2,3,4\}|=4.$
$\phi(10)=|\{x\in\{1,2,3,4,5,6,7,8,9,10\}:gcd(x,10)=1\}|=|\{1,3,7,9\}|=4.$
Hence

$$Z_{c_{10}}=1\cdot a_1^{10}+1\cdot a_2^5+4\cdot a_5^2+4\cdot a_{10}\ .$$

Now, we are in position to find the general expression for D_n.

Theorem 5.1. *Let C_n be the cycle group of permutations generated by $\pi=(1\ 2\ 3.....n)$. Then for each divisor d of n there are $\phi(d)$ permutations in C_n which have n/d cycles of length d, and hence the cycle index of C_n is*

$$Z(C_n)=\frac{1}{n}\sum_{d/n}\phi(d)x_d^{n/d}.$$

Proof. We know by a theorem (if G is a cyclic group of order $n\geq 2$ then for each divisor d of n the number of elements x in G which have order d is $\phi(d)$) that a cyclic group of order n having exactly $\phi(d)$ elements of order d, for each divisor d of n. For this, the $\phi(d)$ permutations are those of the form $\pi^{kn/d}$, where $1\leq k\leq d$ and $gcd(k,d)=1$. Now, only left to show that these permutations have n/d cycles of length d.

Let m be the length of a shortest cycle of the permutation π^i $(1\leq i\leq n-1)$, and suppose x is in a cycle of length m. Then

$$\pi^{im}(x)=(\pi^i)^m(x)=x.$$

For any y in $\{1,2,3,....,n\}$ both x and y are in the single cycle of π, so $y=\pi^r(x)$ for some r. Now

$$(\pi^i)^m(y)=\pi^{im}\pi^r(x)=\pi^r\pi^{im}(x)=\pi^r(x)=y,$$

so that y is in a cycle of π^i whose length divides m. But m is the minimum length of a cycle, so this cycle has length m. Thus all cycles of π^i have the same length m. If the order of π^i is d we must have $d=m$, and so there are n/d cycles of length d as claimed. □

In the study of general problems, the most important tool is a compact notation which records information about the cycle structures of permutations in a group. Let G be a group of permutations of a set X, where frequently we take X to be the set $\{1,2,...,n\}$. Each element g in G can be written in cycle notation with j_i cycles of length i $(1 \le i \le n)$, and the type of g is the corresponding partition

$$[1^{j_1}2^{j_2}...n^{j_n}]$$

of n. Of course, we have $j_1+2j_2+...+nj_n=n$. We shall associate with g an expression

$$Z_g(a_1,a_2,...,a_n)=a_1^{j_1}a_2^{j_2}...\,a_n^{j_n},$$

where the a_i $(1 \le i \le n)$ are, for the moment, simply formal symbols like the x in a polynomial.

5.2 Cycle index of D_n

Theorem 5.2. *The cycle index of D_n is*

$$\frac{1}{2}Z(C_n)+\begin{cases}\frac{1}{4}(a_2^{n/2}+a_1^2a_2^{n/2-1}) & \textit{if n is even,}\\ \frac{1}{2}\,a_1a_2^{(n-1)/2} & \textit{if n is odd.}\end{cases}$$

Proof. Here, we want to prove the case in which mirror-image symmetry is allowed. First, suppose that n is even and $n \ge 4$, and let $n^{'}=\frac{1}{2}n$, so that the corners of n-gon are labeled as $1,2,......,n^{'},n^{'}+1,.....,n$. To the n-gon about the perpendicular bisector of the side $1n$ is the same as taking its 'reflection' in that axis and

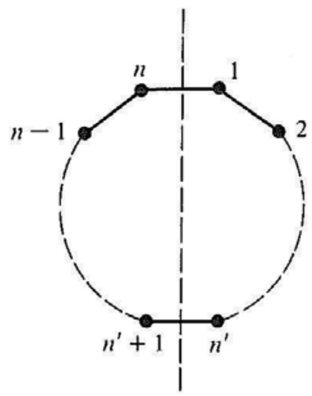

Figure 5.3: Symmetry of an even polygon.

the corresponding permutation is

$$\sigma=(1\,n)(2\,n-1).......(n^{'}\,n^{'}+1).$$

Taking $\pi=(1\;2\;3......n)$ as in previous theorem, we find that

$$\sigma\pi=(1\,n-1)(2\,n-2).......(n^{'}-1\,n^{'}+1)(n^{'})(n),$$

which represents a reflection in the axis $nn^{'}$.

Clearly, there are $n^{'}=\frac{1}{2}n$ reflections in the perpendicular bisectors of the sides; these correspond to the permutations

$$\sigma, \sigma\pi^2, \sigma\pi^4, ..., \sigma\pi^{n-2}.$$

Also, there are $n' = \frac{1}{2}n$ reflections in the axis joining opposite corners, and these correspond to the permutations

$$\sigma\pi, \sigma\pi^3, \sigma\pi^5, ..., \sigma\pi^{n-1}.$$

Thus we have a group of $2n$ permutations, the n rotations π^i and the n reflections $\sigma\pi^i$ $(0 \leq i \leq n-1)$. It is called the dihedral group of order $2n$, and we shall denote it by D_n.

When n is odd, say $n = 2n' + 1$, there are again n reflections, but now they are all of the same type, since each one is a reflection in an axis joining a corner to the mid-point of the opposite side. For example, choosing the corner n' as in figure below, we get

$$\sigma = (1\, n)(2\, n-1).....(n'-1\, n'+1)(n').$$

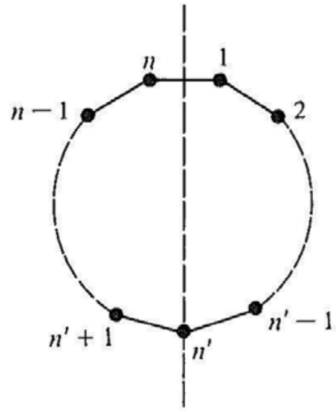

Figure 5.4: Symmetry of an odd polygon.

Here again we have a dihedral group of order $2n$ consisting of the n rotations π^i and the n reflections $\sigma\pi^i$ $(0 \leq i \leq n-1)$.
Now, we come to the final part of the proof.
In the even case, D_n having n elements of C_n, together with $\frac{1}{2}n$ permutations (like σ) of type $[2^{n/2}]$ and $\frac{1}{2}n$ permutations (like $\sigma\pi$) of type $[1^2\, 2^{n/2-1}]$. Hence

$$Z(D_n) = \frac{1}{2n}(\sum_{d/n} \phi(d) a_d^{n/d} + \frac{n}{2} a_2^{n/2} + \frac{n}{2} a_1^2 a_2^{n/2-1}) \quad \text{if } n \text{ is even.} \tag{5.1}$$

which reduces to the form given.
In the odd case we have the n permutations of C_n together with n permutations of type $[1\, 2^{n/2-1}]$. Hence

$$Z(D_n) = \frac{1}{2n}(\sum_{d/n} \phi(d) a_d^{n/d} + n a_1 a_2^{(n-1)/2}) \quad \text{if } n \text{ is odd.} \tag{5.2}$$

From equations (5.1) and (5.2), we have

$$\frac{1}{2}Z(C_n) + \begin{cases} \frac{1}{4}(a_2^{n/2} + a_1^2 a_2^{n/2-1}) & \text{if } n \text{ is even,} \\ \frac{1}{2}\, a_1 a_2^{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

Hence the result. □

5.3 How to use PET to calculate the number of possible colorings of regular n-gons

Polya's theorem provides a mechanical way of computing numbers of inequivalent colorings of various types. In general, the major task is to calculate the cycle index for the relevant group, and it is for this reason that we have prepared ourself with a small list of useful cycle indexes. The secondary task is to expand the expression obtained by substituting for a_i in the cycle index, and hence find the required coefficients.

6 Graphs

6.1 Definitions and Examples

Definition 6.1. A graph G consists of a finite set V, whose members are called **vertices**, and a set E of 2-subsets of V, whose members are called **edges**. We usually write $G = (V,E)$ and say that V is the **vertex-set** and E is the **edge-set**.
For a graph $G(V,E)$, the order of $G = |V|$ and size of $G = |E|$.

Example 6.1. Let the graph on $V = \{A,B,C,D,E\}$ with edge set $E = \{\{A,B\},\{A,E\},\{E,C\},\{B,D\}\}$. Then the order of $G = |V| = 5$ and the size of $G = |E| = 4$.

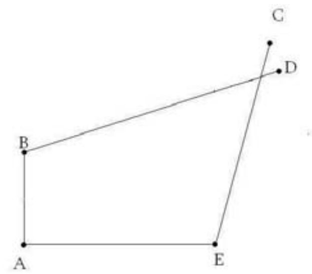

Figure 6.1: Graph of order 5 with size 4.

Definition 6.2. Let $G = (V,E)$ and $G'(V',E')$ be two graphs. We call G and G' **isomorphic**, and write $G \cong G'$, if there exists a bijection $\phi : V \rightarrow V'$ with $\{xy\} \in E \Leftrightarrow \{\phi(x)\phi(y)\} \in E' \;\forall x,y \in V$.
Such a map ϕ is called an **isomorphism** of graphs; if $G = G'$, it is called an **automorphism** of graphs.

Example 6.2. The two graphs in figure 6.2 are isomorphic under the following transformation: $\phi(1)$=A, $\phi(2)$=B, $\phi(3)$=C, $\phi(4)$=D.
The edge lists of both the graphs on the left and the right are (1,2), (1,4), (2,3), (3,4) and $(A,B),(A,D),(B,C),(C,D)$ respectively.

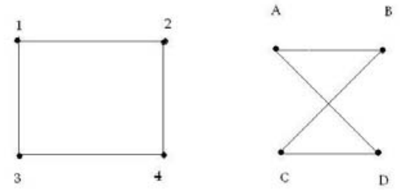

Figure 6.2: Two Isomorphic graphs.

Definition 6.3. As opposed to a multi graph, **a simple graph** is a graph in which each is a pair of distinct vertices and edges do not repeat.

Example 6.3. Let $V = \{1,2,5,7\}$ and let $E = \{\{1,2\},\{1,5\},\{2,5\},\{3,4\},\{5,7\}\}$ we see no unordered pairs are repeated. This shows that it is a simple graph.

Definition 6.4. A **multi graph** is a graph in which each edge is a pair of distinct vertices and edges may repeat.

Example 6.4. Let $V = \{a,b,c,d,e\}$ and let $E = \{\{a,b\},\{a,b\},\{a,e\},\{b,d\},\{c,e\}\}$ we see in the edge set E the first two pairs are repeated. Hence, it is a multi graph.

6.2 Labeled and Unlabeled Graphs

The issue here is whether the names of the vertices matter in deciding whether two graphs are the same. In generation labeled graphs, we seek to construct all possible labellings of all possible graphs topologies. In generating unlabeled graph, we seek only one representative for each topology and ignore labellings.
For example, there are only two connected unlabeled graphs on three vertices- a triangle and a simple path. However, there are four connected labeled graphs on three verticec- one triangle and three 3-vertex paths, each distinguished by the name of their central vertex. In general, labeled graphs are much easier to generate. However, there are so many more of than that we quickly swamped with isomorphic copies of the same few graphs.

Definition 6.5. A **vertex-labeled graph** is a graph in which each vertex is distinguishable from the other by virtue of the underlying system they model. An **edge-labeled graph** can be similarly defined. In an **unlabeled graph** vertices are indistinguishable from each other. It may be possible to distinguish vertices by using the structural properties of the graph. e.g. degree of vertices. Also an unlabeled graph has many labeled graph representations.

6.3 Counting Simple Graphs and Multi Graphs

Definition 6.6. A **labeled graph** is a graph with labels, typically $1,2,.......n$, assigned to the vertices. Two labeled graphs with the same set of labels are considered the same only if there is an isomorphism from one to the other that preserves the labels.

Example 6.5. How many labeled graphs with three vertices?
There are eight different labeled graphs with three vertices as shown in the figure 6.3.

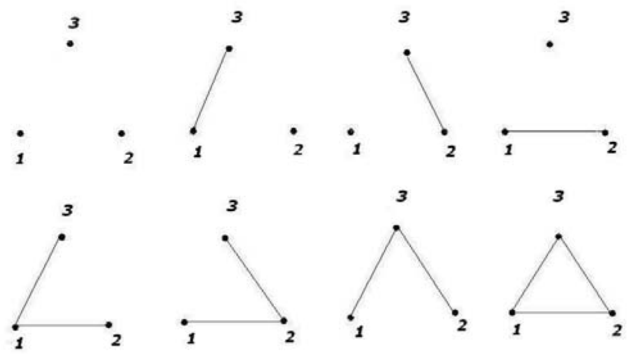

Figure 6.3: All eight labeled graphs on three vertices.

Note: To obtain the number of labeled graphs with n points, we need only observe that each of the $\binom{n}{2}$ possible lines is either present or absent.

Theorem 6.1. *: The number of labeled graphs with n points is* $2^{\binom{n}{2}}$.

Proof. Consider a graph with its vertices labeled $1, 2, \ldots\ldots, n$. In any such graph, each of the $\binom{n}{2}$ possible edges is either present or absent.
Hence the result. □

Example 6.6. How many labeled graphs with two vertices can you construct?
The number of edges with 2 vertices = $\binom{2}{2}$
Hence total number of labeled graphs $= 2^{\binom{2}{2}} = 2^1 = 2$

Theorem 6.2. *How many labeled graphs with n vertices and m edges can we construct?*

Proof. Since the number of distinct edges with n vertices is $\binom{n}{2}$.
Therefore, the number of labeled graphs with n vertices and m edges is the binomial coefficient $\binom{\binom{n}{2}}{m}$.
□

Example 6.7. How many labeled graphs with three vertices and two edges can we construct?
If we put n=3 and m=2 in the theorem 6.2, then number of labeled graphs= $\binom{\binom{3}{2}}{2} = \binom{3}{2} = 3$.
Hence we can construct only 3 labeled graphs with 2 edges.

Example 6.8. How many different unlabeled graphs are there with three vertices?
We can construct only four unlabeled graphs with three vertices, as shown in the figure 6.4.

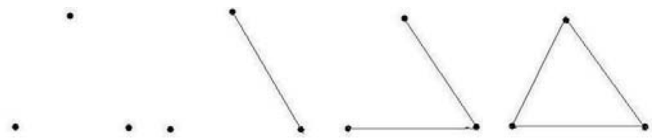

Figure 6.4: All four unlabeled graphs with three vertices.

Definition 6.7. The symmetric group S_n is the group of all permutations acting on the set $X_n = \{1,2,3......n\}$.

Definition 6.8. The pair-permutation α' induced by the permutation α acting on the set X_n is the permutation acting on unordered pairs of elements of X_n defined by the rule $\alpha' : \{x,y\} \to \{\alpha(x),\alpha(y)\}$. The symmetric pair group S_n^2 induced by the symmetric group S_n is the permutation group $\{\alpha' : \alpha \in S_n\}$.

7 Use PET to find non-isomorphic graphs

7.1 Graphical enumeration

It is often important to know how many graphs there are with some desired property. Indeed, any time that graphs are used to model some form of physical structure, the techniques of graphical enumeration are extremely valuable. Many of the techniques for counting graphs are based on the master theorem of George Ploya. Frank Harary and other exploited this master theorem in counting simple graphs, multi graphs, digraph and similar graphical structures.

7.2 Construct a polynomial for counting the number of non-isomorphic graphs

To effectively count graphs, we must define what it means for two graphs to be distinct. The natural definition would be that two graphs G and G' are distinct if there is no permutation of the vertices mapping the edges of G to the edges of G'. The group of permutation on the vertices of a graph of order n is precisely the symmetric group S_n. This group induces a permutation group acting on the edges of K_n in the natural way, which we denote as $S_n^{(2)}$. Distinct graphs, then, are represented by distinct equivalence classes under the action of $S_n^{(2)}$.

7.3 To find cycle structure of different pair group of type $S_n^{(2)}$

Example 7.1. Find the cycle structure of S_3 and $S_3^{(2)}$. Also compare them.
Let $V = \{1,2,3\}$ be the vertex set, then S_v is usually denoted by S_3. Let S_3 be a permutation group for V and $S_3^{(2)}$ permutation group for $V^{(2)}$=$\{(i,j) : i,j \in V, i \neq j\}$.
Then,

$$\left|V^{(2)}\right| = \binom{3}{2} = 3.$$

If we number the elements of $\left|V^{(2)}\right|$ in dictionary order, we get $\{1,2\}\{1,3\}\{2,3\}$.

The group $S_3^{(2)}$is induced by the vertex permutations, then for each $\alpha \in S_3$, there is $\alpha' \in S_3^{(2)}$, such that $\alpha'\{i,j\} = \{\alpha i, \alpha j\}$.
We know that the cycle structure of S_3 is,

$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3)$
$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$
$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$
$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (1)(23)$
$\mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13)(2)$
$\mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)(3)$

Table 7.1: Cycle structure of S_3

Now we find the cycle structure for the elements of $S_3^{(2)}$
Let

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \in S_3$$

Since

$$\rho_0' \in S_3^{(2)}$$

Let

$$\rho_0'\{i,j\} = \{\rho_0 i, \rho_0 j\}$$

So,

$$\rho_0'\{1,2\} = \{1,2\}$$
$$\rho_0'\{1,3\} = \{1,3\}$$
$$\rho_0'\{2,3\} = \{2,3\}$$

This gives,

$$\rho_0' = \begin{pmatrix} 12 & 13 & 23 \\ 12 & 13 & 23 \end{pmatrix} = (12)(13)(23).$$

Similarly, if

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

This gives,

$$\rho_1' = \begin{pmatrix} 12 & 13 & 23 \\ 23 & 12 & 13 \end{pmatrix} = (12\ 23\ 13).$$

Let

$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

This gives,

$$\rho_2' = \begin{pmatrix} 12 & 13 & 23 \\ 13 & 23 & 12 \end{pmatrix} = (12\ 13\ 23).$$

Let

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

This gives,

$$\mu_1{}' = \begin{pmatrix} 12 & 13 & 23 \\ 13 & 12 & 23 \end{pmatrix} = (12\ 13)(23).$$

Let

$$\mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

This gives,

$$\mu_2{}' = \begin{pmatrix} 12 & 13 & 23 \\ 23 & 13 & 12 \end{pmatrix} = (12\ 23)(13).$$

Let

$$\mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

This gives,

$$\mu_3{}' = \begin{pmatrix} 12 & 13 & 23 \\ 12 & 23 & 13 \end{pmatrix} = (12)(13\ 23).$$

Now we can compare cycle structures of S_3 and $S_3^{(2)}$, see Table 7.2.

S_3	Cycle Structure	$S_3^{(2)}$	Cycle Structure
(1)(2)(3)	a_1^3	(12)(13)(23)	a_1^3
(123)	a_3	(12 23 13)	a_3
(132)	a_3	(12 13 23)	a_3
(1)(23)	a_1a_2	(12 13)(23)	a_1a_2
(13)(2)	a_1a_2	(12 23)(13)	a_1a_2
(12)(3)	a_1a_2	(12)(13 23)	a_1a_2

Table 7.2: Comparison between S_3 and S_3^2

We now use only a few basic facts about permutations and permutation groups.The elements of the latter are permutations and the multiplication operation is permutation composition. There must not be any permutations whose product is not in the group.

The cycle index $Z(S)$ of a permutation group S is the average of $\prod_{k=1}^{p} a_k^{j_k}$ over all permutation in the group.

Therefore, the cycle indices of the symmetric group S_3 and the pair group $S_3^{(2)}$ are obtained from the table 7.2.

$$Z(S_3) = Z(S_3^{(2)}) = \frac{1}{6}a_1^3 + \frac{1}{2}a_1a_2 + \frac{1}{3}a_3.$$

The permutation group is $S_3^{(2)}$(pair group acting on the vertices), when we enumerate graphs and the generating function of the objects is $1+z$, indicating whether an edge is present (size one z) or not (1).

Substitute $a_k = 1+z^k$ into $Z(S_3^{(2)})$ to obtain,

$$= \frac{1}{6}(1+z)^3 + \frac{1}{2}(1+z)(1+z^2) + \frac{1}{3}(1+z^3)$$

Expanding by Mathematica we get,

$$\begin{aligned} &= \frac{1}{6}z^3 + \frac{1}{2}z^2 + \frac{1}{2}z + \frac{1}{6} + \frac{1}{2}z^3 + \frac{1}{2}z^2 + \frac{1}{2}z + \frac{1}{2} + \frac{1}{3}z^3 + \frac{1}{3} \\ &= z^3 + z^2 + z + 1 \end{aligned}$$

Which says that there is one graph with three edges, one with two, one with one edge, and one with no edges.
Hence all non-isomorphic graphs on three vertices are four as shown in figure 7.1.

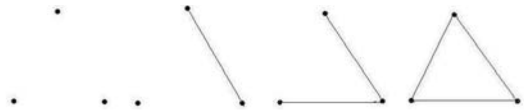

Figure 7.1: All four non-isomorphic graphs on three vertices.

Example 7.2. Find the cycle structure of S_4 and $S_4^{(2)}$. Also compare them.
Let $V = \{1,2,3,4\}$, then S_v is usually denoted by S_4. Let $V^{(2)} = \{a,b,c,d,e,f\}$, then $\left|V^{(2)}\right| = \binom{4}{2} = 6$.
If we number the elements of $V^{(2)}$ in dictionary order, we have,
$a = \{1,2\}, b = \{1,3\}, c = \{1,4\}, d = \{2,3\}, e = \{2,4\}, f = \{3,4\}$.
Then each $\alpha' \in S_v^{(2)}$ can be identified with a permutation in S_6.
Suppose, for example $\alpha = (154) \in S_4$. Then the disjoint cycle factorization of $\alpha' \in S_v^{(2)}$ is computed as follows,

$$\begin{aligned}\alpha'(a) &= \alpha'\{1,2\} = \{\alpha(1),\alpha(2)\} = \{4,2\} = e\\ \alpha'(e) &= \alpha'\{2,4\} = \{\alpha(2),\alpha(4)\} = \{2,3\} = d\\ \alpha'(d) &= \alpha'\{2,3\} = \{\alpha(2),\alpha(3)\} = \{2,1\} = a\end{aligned}$$

So (aed) is one of the cycle in the disjoint cycle factorization of α'.
Continuing,

$$\begin{aligned}\alpha'(b) &= \alpha'\{1,3\} = \{\alpha(1),\alpha(3)\} = \{4,1\} = c\\ \alpha'(c) &= \alpha'\{1,4\} = \{\alpha(1),\alpha(4)\} = \{4,3\} = f\\ \alpha'(f) &= \alpha'\{3,4\} = \{\alpha(3),\alpha(4)\} = \{1,3\} = b\end{aligned}$$

Thus $\alpha' = (aed)(bcf)$.
Similar computation lead to the table 7.3, where each $\alpha' \in S_v^{(2)}$ has been identified with an element of S_6.
Note that $\left|S_v^2\right| = |S_v| = 4! = 24$ a small factor fraction of the $6! = 720$ permutation in S_6.
Since, S_4 and $S_4^{(2)} \subset S_6$ for $V = \{1,2,3,4\}$.
The table 7.3 also shows the comparison between S_4 and $S_4^{(2)}$. The cycle index of the edge permutation group for graphs on four vertices, which has degree six (there are six edges) and order twenty-four (each vertex permutation of the four vertices induces an edge permutation).

α	α'	α	α'
ε_4	ε_6	(142)	(ace)bfd
(12)	(bd)(ce)	(143)	(aed)(bcf)
(13)	(ad)(cf)	(234)	(abc)(dfe)
(14)	(ae)(bf)	(243)	(acb)(def)
(23)	(ab)(ef)	(1234)	(adfc)(be)
(24)	(ac)(df)	(1243)	(aefb)(cd)
(34)	(bc)(de)	(1324)	(af)(bdec)
(13)(24)	(af)(cd)	(1342)	(abfe)(cd)
(123)	(adb)(cef)	(1423)	(af)(bced)
(124)	(aec)(bdf)	(1432)	(acfd)(be)
(132)	(abd)(cfe)	(12)(34)	(be)(cd)
(134)	(ade)(bfc)	(14)(23)	(af)(be)

Table 7.3: Comparison between S_4 and S_4^2

It contains ,

Six 2-cycles
$(12),(13),(14),(23),(24),(34)$
Eight 3-cycles
(123),(124),(134),(234),
(132),(142),(143),(243),
Six 4-cycles
(1234),(1243),(1324),
(1342),(1423),(1432),

This accounts for 20 of the elements of S_4. The remaining 4 elements include the identity element ε_4 which can be thought as a 1-cycle (1) and three elements (13)(24),(12)(34) and (14)(23) which are not cycles.
Therefore, the cycle indices of the symmetric group S_4 and the pair group $S_4^{(2)}$ are obtained from the table 7.3.

$$Z(S_4) = \frac{1}{24}(a_1^4 + 6a_1^2a_2 + 8a_1a_3 + 3a_2^2 + 6a_4). \tag{7.1}$$

From table 7.3, to see how S_4 can be modified to obtain the cycle index polynomial for the pair group $S_4^{(2)}$. Observe that ε_4=ε_6 the polynomial of a_1^4 in S_4 should be replaced with a_1^6. Because from the figure 7.3 the induced action on $V^{(2)}$ of $\alpha = (12)(34)$ is $\alpha' = (be)(cd)$, the term $3a_2^2$ *in* S_4, corresponding to the three permutation in S_4 of cycle type $[2^2]$, is replaced with $3a_1^2a_2^2$. For the same reason, $6a_4$ is replaced with $6a_2a_4$ and $6a_1^2a_2^2$. When $8a_3^2$ is substituted for $8a_1a_3$ and like terms are combined, the transformation of equation (7.2) into equation (7.1) is complete.
Thus, from table 7.3,

$$Z(S_4^{(2)}) = \frac{1}{24}(a_1^6 + 9a_1^2a_2^2 + 8a_3^2 + 6a_2a_4). \tag{7.2}$$

The permutation group is $S_4^{(2)}$(pair group acting on the vertices), when we enumerate graphs and the generation function of the objects is $1+z$, indicating weather an edge is

present (size one z) or not (1).
Substitute $a_k = 1 + z^k$ into $Z(S_4^{(2)})$ to obtain,

$$= \frac{1}{24}((1+z)^6 + 9(1+z)^2(z^2+1)^2 + 8(z^3+1)^2 + 6(z^2+1)(z^4+1))$$

Expanding by Mathematica we get,

$$Z(S_4^{(2)}) = z^6 + z^5 + 2z^4 + 3z^3 + 2z^2 + z + 1$$

Which says that there is one graph with six edges, one with five, two with four edge, three with three edges, two with two edges, one with one edge and one with no edge.
Hence all non-isomorphic graphs on four vertices are 11 as shown in figure 7.2.

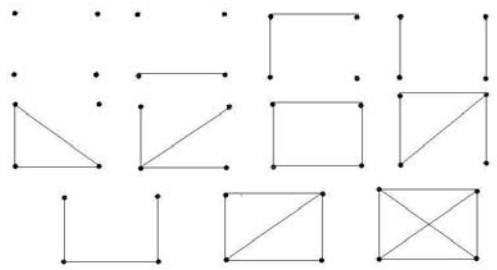

Figure 7.2: All eleven non-isomorphic graphs on four vertices.

Theorem 7.1. *The cycle index of $Z(S_4)$ is,*

$$Z(S_4) = \frac{1}{24}(a_1^4 + 6a_1^2a_2 + 8a_1a_3 + 3a_2^2 + 6a_4).$$

Proof. The 24 vertex-permutation in $Z(S_4)$ are naturally partitioned according to the five possible cycle structures: $a_1^4,\ a_1^2a_2,\ a_1a_3,\ a_2^2,\ a_4$. Each cell in this partition is to be counted.
a_1^4: Only the identity permutation has this cycle structure.
$a_1^2a_2$: There are $\binom{4}{2} = 6$ way to choose two vertices for the 2-cycle.
a_1a_3: There are $\binom{4}{3} = 4$ ways to choose three vertices for the 3-cycle and $(3-1)! = 2$ ways to arrange them in a cycle. Hence there are $4 \times 2 = 8$ permutations with this cycle structure.
a_2^2: There are three ways to group four objects into two cycle, when it does not matter which cycle is written first.
a_4: They correspond to the $(4-1)! = 6$ ways that four objects can be arranged in a cycle. □

Theorem 7.2. *The cycle index of $Z(S_4^{(2)})$ is,*

$$Z(S_4^{(2)}) = \frac{1}{24}(a_1^6 + 9a_1^2a_2^2 + 8a_3^2 + 6a_2a_4).$$

Proof. The cyclic index of the edge permutation group for graphs on four vertices can be find on the light of following five cases. These are the vertex permutation and the edge permutation that they include.

Case 1

The identity. This permutation mapes all vertices (also edges) to themselves and the contribution is a_1^6.

Case 2

Six permutations that exchange two vertices. These permutations preserve the edge that connects the two vertices as well as the edge that connects the two vertices not exchanged. The remaining edges form two two-cycles and the contribution is $6a_1^2a_2^2$.

Case 3

Eight permutations that fix one vertex and produce a three-cycle for the three vertices not fixed. These permutations create two three-cycles of edges, one containing those not incident on the vertices, and another one containing those incident on the vertex and the contribution is $8a_3^2$.

Case 4

Three permutation that exchange two vertex pairs at the same time. These permutation preserve the two edges that connect the two pairs. The remaining edges form two-cycles and the contribution is $3a_1^2a_2^2$.

Case 5

Six permutation that rotate the vertices along a four cycle. These permutations create a 4-cycle of edges and exchange the remaining two edges, the contribution is $6a_2a_4$.

Hence the cycle index of $Z(S_4^{(2)})$ is,

$$Z(S_4^{(2)}) = \frac{1}{24}(a_1^6 + 9a_1^2a_2^2 + 8a_3^2 + 6a_2a_4).$$

□

Enumeration of Multi graphs: Suppose that we are interested in counting multi graphs of n vertices, in which at most two edges are allowed between a pair of vertices. In this case the domain and the permutation group are the same as they were for simple graphs. The range, however, is different. A pair of vertices may be joined 1: no edge, 2: one edge, or 3: two edges. Thus range R contains three elements, say s, t and u with contents, a_0, a_1 and a_2 respectively; that is, a_i indicates the presence of i edges between a vertex pair, for $i = 0,1,2$. Therefore, the figure-counting series becomes
$1 + a + a^2$.
Substituting of $1 + a^r + a^{2r}$ for y_r in $Z(R_n)$ will yield the desired configuration-counting series.

Example 7.3. Find the number of simple multi graphs on four vertices.
Let $S = \{1,2,3,4\}$ and $D = \{present\ once, absent, present\ twice\}$. Let $S_4^{(2)}$ be the pair group on the set of unorder pairs defined on the set S. Again, the cycle index of $S_4^{(2)}$ is,

$$Z(S_4^{(2)})(a_1, \ldots\ldots, a_4) = \frac{1}{24}(a_1^6 + 9a_1^2a_2^2 + 8a_3^2 + 6a_2a_4).$$

Clearly,

$$f(a) = 1 + a + a^2.$$

So, the figure-counting series is given by replacing a_i by,

$$f(z^i) = 1 + z^i + z^{2i}.$$

Hence, we have,

$$\begin{aligned}Z(S_4^{(2)}) =&\frac{1}{24}[(1+z+z^2)^6+9(1+z+z^2)^2(1+z^2+z^4)^2+8(1+z^3+z^6)^2+\\&6(1+z^2+z^4)(1+z^4+z^8)].\\=&1+z+3x^2+5z^3+8z^4+9z^5+12z^6+9z^7+8z^8+5z^9+3z^{10}+z^{11}+z^{12}.\end{aligned}$$

Again, the coefficient of a^j for $0 \leq j \leq 12$ is the number of simple graphs on four vertices with j edged.

7.4 Find cyclic index or polynomial for the pair group induced by S_n.

Lemma 7.3. *The cycle index of the symmetric group S_n is given by*

$$Z(S_n) = \sum_{(j)} \frac{1}{\prod_{k=1}^{n} j_k! k^{j_k}} \prod a_k^{j_k(g)}$$

Where the summation is taken over all partitions (j) of n.

Proof. Consider some partition (j) of n, where $(j) = (j_1, j_2, ...j_n)$. Assume that the cycles of some permutation having cycles given by (j) are order from largest to smallest. The n elements of the object set can be order in $n!$. different ways. However, for each k, the j_k cycles can be ordered in $j_k!$ different ways, and can begin in k different elements. Thus, any permutation is represented $\prod a_k^{j_k(g)}!$ times, so that there are a total of $\frac{n!}{\prod_{k=1}^{p} j_k! k^{j_k}}$ permutation with cycle structure given by (j). This allows us to re-index over (j) rather than over the individual permutation obtaining the cycle structure as stated in the theorem. □

The cycle index polynomial $Z(S_p^{(2)})$ used for counting simple graph by Harry [5]. It is

$$Z(S_p^{(2)}) = \frac{1}{p!} \sum_{(j)} \frac{p!}{\prod_{k=1}^{p} j_k! k^{j_k}} \prod_{k=1}^{[p/2]} (a_k a_{2k}^{k-1})^{j_{2k}} \prod_{k=0}^{[p-1/2]} a_{2k+1}^{k j_{2k+1}} \prod_{k=1}^{[p/2]} a_k^{k(2^{jk})} \prod_{1 \leq r < s \leq p-1} a_{lcm(r,s)}^{gcd(r,s) j_r j_s}$$

Here $gcd(r,s)$ and $lcm(r,s)$ are the least common multiple and greatest common deviser or r and s, respectively. We consider edges between cycles of different lengths r and s, with $r < s$. There are rs such edges, and each edge lies in an induced cycles of length $[r,s]$. Each vertex must be mapped to itself,so that r and s must both divide the length of this cycle,so that $[r,s]$ is a lower bound on the length of such a cycle. On the other hand, at $[r,s]$ applications of α, such an edge is mapped to itself. Since each cycle has length $[r,s]$, there must be exactly (r,s) such cycles for each choice of r and s. Clearly, there are $j_r j_s$ choices of r and s, so that the contribution for given values of r and s is $a_{lcm(r,s)}^{gcd(r,s) j_r j_s}$. Also j denotes summation for $1j_1 + 2j_2 + ... + nj_n = n$. From the product $\prod_{k=0}^{[p-1/2]} a_{2k+1}^{k j_{2k+1}}$ we see that, with one exception, $j_{2k+1} = 0$, since otherwise f_{2k+1} occurs. The exception is that we may have $4j_1 = 1$, since in this case the power of a_1 is zero. From the product,

$$\prod_{k=1}^{[p/2]} a_k^k (2^{j_k})(a_k a_{2k}^{k-1})^{j_{2k}}.$$

We see that $j_{2_k} = 0$ if k is odd, since otherwise a_k occurs. So for non zero terms we have $J_i = 0$ unless $i = 1$ (in which case $J_1 = 1$).

Example 7.4. Write down the cycle index for the pair group $Z(S_p^{(2)})$.
Polya enumeration involves permutations (j) of the set $X_n = \{1,2,....n\}$ j_k denotes the number of k-cycles in (j) for $k = 1,2,.....n$.
For example, if $(j) = (12)(34)(567)$ then $j_2 = 2,\ j_3 = 1, j_1 = j_4 = j_5 = j_6 = j_7 = 0$.
Now, for 5 vertices, we obtain the following option for,

$$
\begin{aligned}
(j) =&(j_1, j_2, j_3, j_4, j_5).\\
&(0,0,0,0,1)(1,0,0,1,0)\\
&(0,1,1,0,0)(2,0,1,0,0)\\
&(1,2,0,0,0)(3,1,0,0,0)\\
&(5,0,0,0,0).
\end{aligned}
$$

Then give the following summands.
For $j = (0,0,0,0,1)$, we have

$$\frac{1}{5^1 1!}(a_5)^2(1)(1)(1) = \frac{1}{5}a_5^2$$

For $j = (1,0,0,1,0)$, we have

$$\frac{1}{4^1 1! 1^1 1!}(1)(a_2a_4)(1)(a_4) = \frac{1}{4}a_2a_4^2.$$

For $j = (0,1,1,0,0)$, we have

$$\frac{1}{3!1!2^1 1!}(a_3)(a_1a_2^0)(1)(a_6) = \frac{1}{6}a_1a_3a_6.$$

For $j = (2,0,1,0,0)$, we have

$$\frac{1}{3^1 1! 1^2 2!}(a_3)(1)(a_1)(a_3^2) = \frac{1}{6}(a_1)(a_3^3).$$

For $j = (1,2,0,0,0)$, we have

$$\frac{1}{1^1 1! 2^2 2!}(1)(a_1a_2^0)^2(a_2^2)(a_2^2) = \frac{1}{8}a_1^2a_2^4.$$

For $j = (3,1,0,0,0)$, we have

$$\frac{1}{1^3 3! 2^1 1!}(1)(a_1a_2^0)(a_1^3)(a_2^3) = \frac{1}{12}a_1^4a_2^3.$$

For $j = (5,0,0,0,0)$, we have

$$\frac{1}{1^5 5!}(1)(1)(a_1^{10})(1) = \frac{1}{120}a_1^{10}.$$

Therefor, the cycle index of the pair group $Z(S_p^{(2)})$ is,

$$Z(S_p^{(2)}) = \frac{1}{120}(a_1^{10} + 10a_1^4a_2^3 + 20a_1a_3^3 + 15a_1^2a_2^4 + 30a_2a_4^2 + 20a_1a_3a_6 + 24a_5^2).$$

Summing and substituting $(1+z^k)$ for a_k yields.

$$\begin{aligned}Z(S_5^2) =&\frac{1}{5}(1+z^5)^2+\frac{1}{4}(1+z^2)(1+z^4)^2+\frac{1}{6}(1+z)(1+z^3)(1+z^6)+\\&\frac{1}{6}(1+z)(1+z^3)^3+\frac{1}{8}(1+z)^2(1+z^2)^4+\\&\frac{1}{12}(1+z)^4(1+z^2)^3+\frac{1}{120}(1+z)^{10}.\end{aligned}$$

By using Mathematica, we have

$$1+z+2z^2+4z^3+6z^4+6z^5+6z^6+4z^7+2z^8+z^9+z^{10}.$$

So there is one graph on five vertices with each of zero, one, nine, or ten edges, two graphs with each of two or eight edges, for graph with each of 3 or 7 edges and 6 graphs with each of 5,6, or 7 edges for a table of 34 graphs on 5 vertices.

7.5 Discussion and Conclusion

In section 6 and 7 of this thesis, we only talked about simple, labeled and unlabeled graphs. We discussed a way for finding these graphs on three, four, and five vertices to use PET and took a look at some theorem and examples. We concerned with the algorithmics rather than the mathematics. For the application to counting the non-isomorphic types of the n-vertex simple graphs, we can determine the number of non-isomorphic types having each possible number of edges. Polya Enumeration Theorem, provides an elegant method for determining the number of non-isomorphic graphs. Generally, the major task is to find the cycle index for the relevant group, and that is the reason that we have armed ourself with a small list of useful cycle indexes. The secondary task is to expand the expression obtained by substituting for a_i in the cycle index, and hence find the required coefficients.

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