Lecture 3: Fourier Series: pointwise and uniform convergence.

1. Introduction.

At the end of the second lecture we saw that we had for each function \( f \in L^2([-\pi, \pi]) \) a Fourier series

\[
f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),
\]

where the coefficients \( a_k, b_k \) are defined as

\[
a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx,
\]

with \( k = 0, 1, \ldots \) (note that this means that \( b_0 = 0 \)). This is called the real form of the Fourier series.

The two objects, \( f \) and the series, are not always equal at each point \( x \in [-\pi, \pi] \), but we noted that they are equal as vectors in \( L^2([-\pi, \pi]) \). In this lecture we shall look at conditions which allow us to put

\[
f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).
\]

To this end, we define two spaces of functions which are contained in \( L^2([-\pi, \pi]) \).

Definition 1.1. We define the space \( E \) as the space of piecewise continuous functions on \([-\pi, \pi]\).

The function \( f(x) \) is piecewise continuous on an interval \( I \) if it is continuous on \( I \) except perhaps for a finite number of points, and if \( a \in I \) is a point of discontinuity for \( f(x) \) then \( f(a_+) \) and \( f(a_-) \) exist: that is

\[
f(a_+) = \lim_{x \to a_+} f(x), \quad f(a_-) = \lim_{x \to a_-} f(x)
\]

are required to exist.

Example 1.1. The function \( f(x) \) defined as

\[
f(x) = \begin{cases} 
  e^x & \text{for } x < 0, \\
  2 - x & \text{for } 0 \leq x \leq 2, \\
  x^2 & \text{for } x > 2.
\end{cases}
\]

To see this, we note that the points of discontinuity are \( x = 0 \) and \( x = 2 \). Then we easily calculate that \( f(0_-) = 1, f(0_+) = 2 \) and \( f(2_-) = 0, f(2_+) = 4 \).

Then we build upon this definition to define another space \( E' \) as follows:
Definition 1.2. The space $E'$ is defined as the space of all functions $f(x) \in E$ such that the right-hand derivative $D_+ f(x)$ exists for all $-\pi \leq x < \pi$, and the left-hand derivative $D_- f(x)$ exists for all $-\pi < x \leq \pi$. Recall that

$$D_+ f(x) = \lim_{h \to 0^+} \frac{f(x + h) - f(x_+)}{h}$$

$$D_- f(x) = \lim_{h \to 0^-} \frac{f(x + h) - f(x_-)}{h}.$$ 

Note that we need to know $f(x_+)$ in order to calculate $D_+ f(x)$, and $f(x_-)$ in order to calculate $D_- f(x)$.

Remark 1.1. Every continuous function is piecewise continuous, that is, belongs to $E$. Every function which is differentiable on $[-\pi, \pi]$ belongs to $E'$. All functions in $E$ belong to $L^2([-\pi, \pi])$.

2. Pointwise convergence of Fourier series.

We shall prove that the Fourier series of a given function converges pointwise to that function provided the function is continuous and belongs to $E'$. But first we need some notation: we write $S_N(x)$ for

$$\frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx).$$

Our result is:

**Theorem 2.1.** If $f \in E'$ and

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

then the series converges pointwise to

$$\frac{f(x_+) + f(x_-)}{2}.$$ 

That is

$$S_N(x) \to \frac{f(x_+) + f(x_-)}{2} \text{ as } N \to \infty.$$ 

In particular, we have

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

at every point $x \in [-\pi, \pi]$ where $f(x)$ is continuous.

Theorem 2.1 tells us about the pointwise convergence of the Fourier series to the value of the function at points where the function is continuous, and to the mean value of the left- and right-hand limits of the function at points of (finite) discontinuity. However, we would also like to know when we have uniform convergence. This will give us the possibility of integrating the series term-wise. To this end, we have the following result:
Theorem 2.2. Suppose that

- \( f(t) \) is continuous on \([-\pi, \pi]\)
- \( f(-\pi) = f(\pi) \)
- \( f'(t) \in E \)

Then

\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)
\]

converges uniformly to \( f(x) \) on \([-\pi, \pi]\).

We leave the proofs of these theorems to the Appendix of these notes.


In this section we take up the definitions of even and odd functions:

Definition 3.1. \( f(t) \) is said to be even if

\[ f(-t) = f(t) \text{ for all } t \in \mathbb{R}. \]

\( f(t) \) is said to be odd if

\[ f(-t) = -f(t) \text{ for all } t \in \mathbb{R}. \]

Elementary examples are: \( f(t) = \cos t \) is even, and \( f(t) = \sin t \) is odd. Not all functions are of these two types, but they are a mixture of them. More precisely:

Proposition 3.1. Each function \( f(t) \) can be written as

\[
f(t) = F_+(t) + F_-(t)
\]

where \( F_+(t) \) is even and \( F_-(t) \) is odd.

Proof: Put

\[
F_+(t) = \frac{f(t) + f(-t)}{2}, \quad F_-(t) = \frac{f(t) - f(-t)}{2}.
\]

it is elementary to verify that \( F_+(t) \) is even and that \( F_-(t) \) is odd.

useful results are:

Proposition 3.2. (1) \( f(t), g(t) \) even \( \implies \) \( f(t)g(t) \) is even.

(2) \( f(t), g(t) \) odd \( \implies \) \( f(t)g(t) \) is even.

(3) \( f(t) \) even and \( g(t) \) odd \( \implies \) \( f(t)g(t) \) is odd.
The proof of this is an elementary verification. We also record the results:

$$\int_{-a}^{a} f(t)dt = 0$$

if \( f(t) \) is odd (and integrable, of course), whereas we have

$$\int_{-a}^{a} f(t)dt = 2 \int_{0}^{a} f(t)dt$$

if \( f(t) \) is even (and integrable).

Note that the Fourier series of an odd function contains only sine-terms, whereas the Fourier series of an even function contains only cosine-terms.

Now we come to another definition:

**Definition 3.2.** If \( f(t) \) is defined on \([0, \pi]\), then the function \( F(t) \) defined by

\[
F(t) = \begin{cases} 
  f(t) & \text{if } 0 \leq t \leq \pi \\
  f(-t) & \text{if } -\pi \leq t < 0,
\end{cases}
\]

is called the **even extension** of \( f(t) \).

The function \( G(t) \) defined by

\[
G(t) = \begin{cases} 
  f(t) & \text{if } 0 < t \leq \pi \\
  0 & t = 0 \\
  -f(-t) & \text{if } -\pi \leq t < 0,
\end{cases}
\]

is called the **odd extension** of \( f(t) \).

As we have noted above, the Fourier series of an even function contains only cosine-terms, and the Fourier series of an odd function contains only sine-terms. This then gives us the following definition:

**Definition 3.3.** The **cosine series** of \( f(t) \) defined on \([0, \pi]\) is the Fourier series of the even extension of \( f(t) \).

The **sine series** of \( f(t) \) defined on \([0, \pi]\) is the Fourier series of the odd extension of \( f(t) \).

4. **Solving ordinary differential equations with Fourier series.**

In this section I want to show how to solve differential equations using Fourier series. First, note that there is one restriction: the function(s) in the differential equation must be periodic. In our case, we look at functions with period \( 2\pi \), but the theory holds for any other period.

An important fact is that if \( y(t) \) has period \( T \) then its derivative \( y'(t) \) also has period \( T \) (provided it exists). This is easy to prove:

**Lemma 4.1.** If \( y(t) \) has period \( T \) and is differentiable, then \( y'(t) \) also has period \( T \).
Proof: We know that $y(t + T) = y(t)$ for all $t \in \mathbb{R}$ and we want to prove that $y'(t + T) = y'(t)$. We have

$$y'(t + T) = \lim_{h \to 0} \frac{y(t + T + h) - y(t + T)}{h}$$

$$= \lim_{h \to 0} \frac{y(t + h) - y(t)}{h}$$

$$= y'(t).$$

Another result that we need is the following:

**Lemma 4.2.** If $y(t)$ has period $2\pi$ and is differentiable and has complex Fourier series

$$y(t) \sim \sum_{-\infty}^{\infty} c_k e^{ikt},$$

then

$$y'(t) \sim \sum_{-\infty}^{\infty} ikc_k e^{ikt}.$$  

In other words, if $c_k$, $k = \ldots, -2, -1, 0, 1, 2,$ ... are the Fourier coefficients of $y(t)$, then the Fourier coefficients of $y'(t)$ are $ikc_k$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$.

**Proof:** Suppose we have

$$y'(t) \sim \sum_{-\infty}^{\infty} d_k e^{ikt}.$$  

Then we have

$$d_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} y'(t)e^{-ikt} dt$$

$$= \left[ y(t)e^{-ikt} \right]_{-\pi}^{\pi} + \frac{1}{2\pi} \int_{-\pi}^{\pi} (ik)y(t)e^{-ikt} dt$$

$$= ikc_k$$

because we have

$$[y(t)e^{-ikt}]_{-\pi}^{\pi} = 0$$

since both $y(t)$ and $e^{-ikt}$ have period $2\pi$ and because

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(t)e^{-ikt} dt.$$  

We now look at an example.

**Example:** Find all functions $y(t)$ with period $2\pi$ and which satisfy the differential equation

$$y''(t) - 2y(t + \pi) = \cos t.$$
Solution: First note that \( y''(t) \) exists and so both \( y'(t) \) and \( y(t) \) are continuous, since if \( f'(t) \) exists at \( t \), then \( f(t) \) is continuous at \( t \). Note also that we then have

\[
y''(t) = y(t + \pi) + \cos t
\]

so that \( y''(t) \) is also continuous, because both \( y(t) \) and \( \cos t \) are continuous. Since both these functions are differentiable and because their derivatives are continuous, then \( y''(t) \) is differentiable and \( y'''(t) \) is continuous.

From the above reasoning, we conclude that \( y(t) \), \( y'(t) \) and \( y''(t) \) all satisfy the conditions of Dirichlet’s theorem, and so we may put them equal to their Fourier series. So if we have

\[
y(t) = \sum_{-\infty}^{\infty} c_k e^{ikt}
\]

then

\[
y'(t) = \sum_{-\infty}^{\infty} ik c_k e^{ikt}.
\]

and

\[
y''(t) = \sum_{-\infty}^{\infty} (ik)^2 c_k e^{ikt}.
\]

Further,

\[
y(t + \pi) = \sum_{-\infty}^{\infty} c_k e^{ik(t+\pi)} = \sum_{-\infty}^{\infty} e^{ik\pi} c_k e^{ikt} = \sum_{-\infty}^{\infty} (-1)^k c_k e^{ikt}
\]

since

\[
e^{ik\pi} = (e^{i\pi})^k = (-1)^k.
\]

Putting all this into the differential equation, we have (by rewriting \( \cos t \) in complex form)

\[
\sum_{-\infty}^{\infty} (ik)^2 c_k e^{ikt} - 2 \sum_{-\infty}^{\infty} (-1)^k c_k e^{ikt} = \frac{e^{it}}{2} + \frac{e^{-it}}{2},
\]

which gives the equation

\[
\sum_{-\infty}^{\infty} [(ik)^2 - 2(-1)^k] c_k e^{ikt} = \frac{e^{it}}{2} + \frac{e^{-it}}{2}.
\]

From this equation, it follows that


\[(ik)^2 - 2(-1)^k c_k = 0\]

for all \( k \neq \pm 1 \). The reasoning behind this step is that the functions \( \{e^{ikt}, k = 0, \pm 1, \pm 2, \pm 3, \ldots \} \) form a basis for \( L^2([\pi, \pi]) \), and that coefficients of basis vectors in the left-hand side of an equation are equal to the corresponding coefficients in the right-hand side of the equation. We then have that either \( c_k = 0 \) or \( (ik)^2 - 2(-1)^k = 0 \) for these \( k \). Now

\[(ik)^2 - 2(-1)^k = 0\]

gives

\[k^2 = -2(-1)^k\]

and this has no integer solution: for even \( k \) we have \( k^2 = -2 \), which is impossible; for odd \( k \) we have \( k^2 = 2 \) which is also impossible because \( \sqrt{2} \) is irrational. Thus we must have \( (ik)^2 - 2(-1)^k \neq 0 \) for all \( k \neq \pm 1 \) and hence \( c_k = 0 \) for \( k \neq \pm 1 \).

For \( k = \pm 1 \) we have the following:

\[-3c_1 = \frac{1}{2}, \quad -3c_{-1} = \frac{1}{2}\]

so that

\[c_{-1} = -\frac{1}{6}, \quad c_1 = -\frac{1}{6},\]

and so we have

\[y(t) = -\frac{1}{3} \cos t.\]
5. Appendix: Proofs of pointwise and uniform convergence.

Theorem 5.1. If \( f \in E' \) and

\[ f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \]

then the series converges pointwise to

\[ \frac{f(x_+) + f(x_-)}{2} . \]

That is

\[ S_N(x) \to \frac{f(x_+) + f(x_-)}{2} \quad \text{as} \quad N \to \infty. \]

In particular, we have

\[ f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \]

at every point \( x \in [-\pi, \pi] \) where \( f(x) \) is continuous.

Proof: The proof is quite lengthy, so we break it up into a series of Lemmas.

The first step is to prove the result

Lemma 5.1.

\[ a_k \cos kx + b_k \sin kx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \cos kt \, dt \]

Proof: We have, using the definition of the constants \( a_k, b_k, \)

\[ a_k \cos kx + b_k \sin kx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) [\cos kt \cos kx + \sin kt \sin kx] \, dt \]

\[ = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos k(t - x) \, dt \]

\[ = \frac{1}{\pi} \int_{-(\pi-x)}^{\pi-x} f(s + x) \cos k s \, ds \]

\[ = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s + x) \cos k s \, ds \]

where we have first used the substitution \( s = t - x \), and then used the result that if \( F(s) \) has period \( 2\pi \) then

\[ \int_{\alpha}^{\alpha+2\pi} F(s) ds = \int_{-\pi}^{\pi} F(s) ds \]

for all \( \alpha \): in fact, this result with \( F(s) = f(s + x) \cos ks \) and \( \alpha = -(\pi - x) \) gives us

\[ \frac{1}{\pi} \int_{-(\pi-x)}^{\pi-x} f(s + x) \cos ks \, ds = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s + x) \cos ks \, ds . \]

Thus we have
\[ a_k \cos kx + b_k \sin kx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t + x) \cos kt \, dt. \]

This result and the fact that
\[ \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + t) \, dt, \]
which is easy to verify, when substituted into \( S_N(x) \) (as defined above) give
\[
S_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \left( \frac{1}{2} + \sum_{k=1}^{N} \cos kt \right) \, dt \\
= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) D_N(t) \, dt,
\]
where \( D_N(t) \) is defined as
\[
D_N(t) = \frac{1}{2} + \sum_{k=1}^{N} \cos kt,
\]
which bears the name **Dirichlet’s kernel**.

The aim is to prove that
\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) D_N(t) \, dt \to \frac{f(x_+) + f(x_-)}{2} \quad \text{as } N \to \infty.
\]
We do this by proving
\[
\frac{1}{\pi} \int_{0}^{\pi} f(x + t) D_N(t) \, dt \to \frac{f(x_+)}{2}
\]
and
\[
\frac{1}{\pi} \int_{-\pi}^{0} f(x + t) D_N(t) \, dt \to \frac{f(x_-)}{2}
\]
as \( N \to \infty \).

The next step is to prove the formula
**Lemma 5.2.**
\[
D_N(t) = \frac{\sin(N + \frac{1}{2})t}{2 \sin \frac{t}{2}} \quad \text{for } t \neq 0.
\]
**Proof:** We have first that
\[ D_N(t) = \frac{1}{2} + \sum_{k=1}^{N} \cos kt \]

\[ = \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{N} [e^{ikt} + e^{-ikt}] \]

\[ = \frac{1}{2} \left( 1 + \sum_{k=1}^{N} e^{ikt} + \sum_{k=1}^{N} e^{-ikt} \right). \]

Then note that \( \sum_{k=1}^{N} q^k = \frac{1 - q^{N+1}}{1 - q} \) whenever \( q \neq 1 \), so if \( t \neq 0 \) we put \( q = e^{it} \) to obtain

\[ \sum_{k=1}^{N} e^{ikt} = \frac{e^{it} - e^{(N+1)it}}{1 - e^{it}} \]

\[ = \frac{e^{it} - e^{(N+1)it}}{e^{\frac{it}{2}}(e^{\frac{it}{2}} - e^{-\frac{it}{2}})} \]

\[ = \frac{e^{\frac{it}{2}} - e^{(N+\frac{1}{2})it}}{2i \sin \frac{it}{2}}, \]

and then we take \( q = e^{-it} \) to obtain

\[ \sum_{k=1}^{N} e^{-ikt} = \frac{e^{-it} - e^{-(N+1)it}}{1 - e^{-it}} \]

\[ = \frac{e^{-it} - e^{-(N+1)it}}{e^{-\frac{it}{2}}(e^{\frac{it}{2}} - e^{-\frac{it}{2}})} \]

\[ = \frac{e^{-\frac{it}{2}} - e^{-(N+\frac{1}{2})it}}{2i \sin \frac{it}{2}}. \]

Combining these two calculations, we find

\[ \sum_{k=1}^{N} e^{ikt} + \sum_{k=1}^{N} e^{-ikt} = \frac{1}{2i \sin \frac{it}{2}} \left[ e^{-\frac{it}{2}} - e^{\frac{it}{2}} + e^{(N+\frac{1}{2})it} + e^{-(N+\frac{1}{2})it} \right] \]

\[ = \frac{1}{2i \sin \frac{it}{2}} \left[ -2i \sin \frac{it}{2} + 2i \sin \left( N + \frac{1}{2} \right)it \right] \]

\[ = -1 + \frac{\sin \left( N + \frac{1}{2} \right)it}{\sin \frac{it}{2}} \]

and from this it follows that

\[ D_N(t) = \frac{\sin \left( N + \frac{1}{2} \right)it}{\sin \frac{it}{2}} \]
for $t \neq 0$. We can calculate $D_N(0)$: we have $D_N(0) = N + \frac{1}{2}$, and it is easy to verify that $D_N(t) \to D_N(0)$ as $t \to 0$, so we may write

$$D_N(t) = \begin{cases} \frac{\sin(N+\frac{1}{2})}{\sin \frac{t}{2}}, & t \neq 0 \\ N + \frac{1}{2}, & t = 0. \end{cases}$$

Then we define two functions $g(t)$ and $h(t)$ by

$$g(t) = \frac{f(x + t) - f(x_+)}{2 \sin \frac{t}{2}}, \quad h(t) = \frac{f(x + t) - f(x_-)}{2 \sin \frac{t}{2}}.$$

The function $g(t)$ is piecewise continuous on $[0, \pi]$ and $h(t)$ is piecewise continuous on $[-\pi, 0]$. We find that

$$\lim_{t \to 0^+} g(t) = \lim_{t \to 0^+} \frac{f(x + t) - f(x_+)}{t} \cdot \frac{t}{2 \sin \frac{t}{2}} = D_+ f(x)$$

as well as

$$\lim_{t \to 0^-} h(t) = \lim_{t \to 0^-} \frac{f(x + t) - f(x_-)}{t} \cdot \frac{t}{2 \sin \frac{t}{2}} = D_- f(x).$$

The limits exist because $f \in E'$. Finally, using these results, we calculate that

$$\lim_{N \to \infty} \frac{1}{\pi} \int_0^\pi f(x + t) D_N(t) \, dt = \frac{f(x_+)}{2}$$

and that

$$\lim_{N \to \infty} \frac{1}{\pi} \int_{-\pi}^0 f(x + t) D_N(t) \, dt = \frac{f(x_-)}{2}.$$

From all this follows the result:

$$\lim_{N \to \infty} S_N(x) = \frac{1}{\pi} \int_{-\pi}^\pi f(x + t) D_N(t) \, dt = \frac{f(x_+) + f(x_-)}{2},$$

as claimed. The rest of the theorem follows easily. To show that these limits hold, we have:

**Lemma 5.3.**

$$\lim_{N \to \infty} \frac{1}{\pi} \int_0^\pi f(x + t) D_N(t) \, dt = \frac{f(x_+)}{2}$$

and

$$\lim_{N \to \infty} \frac{1}{\pi} \int_{-\pi}^0 f(x + t) D_N(t) \, dt = \frac{f(x_-)}{2}.$$
**Proof:** An elementary calculation using the definition of $D_N(t)$ gives

$$
\int_0^\pi D_N(t) dt = \frac{\pi}{2}, \quad \int_{-\pi}^0 D_N(t) dt = \frac{\pi}{2}
$$

Then we have

$$
\frac{1}{\pi} \int_0^\pi f(x + t) D_N(t) dt - \frac{f(x_+)}{2} = \frac{1}{\pi} \int_0^\pi f(x + t) D_N(t) dt - \frac{1}{\pi} \int_0^\pi f(x_+ D_N(t) dt
$$

$$
= \frac{1}{\pi} \int_0^\pi (f(x + t) - f(x_+)) D_N(t) dt
$$

$$
= \frac{1}{\pi} \int_0^\pi 2 \sin t g(t) D_N(t) dt
$$

$$
= \frac{1}{\pi} \int_0^\pi g(t) \sin \left( N + \frac{1}{2} \right) t dt.
$$

Now, by the **Riemann-Lebesgue Lemma** (see Lecture 2), we find that

$$
\int_0^\pi g(t) \sin \left( N + \frac{1}{2} \right) t dt \to 0 \quad \text{as} \quad N \to \infty.
$$

Consequently,

$$
\frac{1}{\pi} \int_0^\pi f(x + t) D_N(t) dt - \frac{f(x_+)}{2} \to 0 \quad \text{as} \quad N \to \infty,
$$

or

$$
\lim_{N \to \infty} \frac{1}{\pi} \int_0^\pi f(x + t) D_N(t) dt = \frac{f(x_+)}{2}
$$

To show that

$$
\int_0^\pi g(t) \sin \left( N + \frac{1}{2} \right) t dt \to 0 \quad \text{as} \quad N \to \infty;
$$

we put

$$
G(t) = \begin{cases} 
    g(t) & \text{if} \ 0 < t \leq \pi \\
    0 & \text{if} \ -\pi \leq t \leq 0.
\end{cases}
$$

Then, since $g(t)$ is piecewise continuous on $[0, \pi]$, it follows that $G(t)$ is piecewise continuous on $[-\pi, \pi]$, and hence $G(t) \in L^2([-\pi, \pi])$ so that the Riemann-Lebesgue Lemma gives

$$
\int_0^\pi g(t) \sin \left( N + \frac{1}{2} \right) t dt = \int_{-\pi}^\pi G(t) \sin \left( N + \frac{1}{2} \right) t dt \to 0 \quad \text{as} \quad N \to \infty.
$$

A similar calculation on $8 - \pi, 0]$ using the function $h(t)$ defined above gives us

$$
\lim_{N \to \infty} \frac{1}{\pi} \int_{-\pi}^0 f(x + t) D_N(t) dt = \frac{f(x_-)}{2}.
$$

We leave the details to you.
Theorem 2.1 tells us about the pointwise convergence of the Fourier series to the value of the function at points where the function is continuous, and to the mean value of the left- and right-hand limits of the function at points of (finite) discontinuity. However, we would also like to know when we have uniform convergence. This will give us the possibility of integrating the series term-wise. To this end, we have the following result:

**Theorem 5.2.** Suppose that
- $f(t)$ is continuous on $[-\pi, \pi]$
- $f(-\pi) = f(\pi)$
- $f'(t) \in E$

Then

$$a_0^2 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

converges uniformly to $f(x)$ on $[-\pi, \pi]$.

**Proof:** Since we have $f'(t) \in E$, we have the Fourier series

$$f'(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (\alpha_k \cos kt + \beta_k \sin kt)$$

and, with

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt),$$

it is easy to calculate (using the definition of the coefficients in a Fourier expansion) that $\alpha_0 = 0$ (this is guaranteed by $f(-\pi) = f(\pi)$) and that $\alpha_k = kb_k$, $\beta_k = -ka_k$ so we find that

$$f'(t) \sim \sum_{k=1}^{\infty} (kb_k \cos kt - ka_k \sin kt).$$

Now, Parseval’s identity (see Lecture 2) guarantees that

$$\sum_{k=1}^{\infty} (|\alpha_k|^2 + |\beta_k|^2) = \|f'\|^2 < +\infty,$$

so that

$$\sum_{k=1}^{\infty} (k^2|a_k|^2 + k^2|b_k|^2)$$

converges. We shall show that the series

$$\sum_{k=1}^{\infty} \sqrt{|a_k|^2 + |b_k|^2}$$

also converges. First consider, for each $N \geq 1$, the vectors
\[ u_N = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N} \right) \]

and

\[ v_N = (\sqrt{\alpha_1^2 + \beta_1^2}, \sqrt{\alpha_2^2 + \beta_2^2}, \ldots, \sqrt{\alpha_N^2 + \beta_N^2}) \]

Then we have

\[
\sum_{k=1}^{N} \sqrt{|a_k|^2 + |b_k|^2} = \sum_{k=1}^{N} \frac{1}{k} \sqrt{|\alpha_k|^2 + |\beta_k|^2} \\
= u_N \cdot v_N \\
\leq \|u_N\|\|v_N\| \text{ by the Cauchy-Schwarz Theorem} \\
= \sqrt{\sum_{k=1}^{N} \frac{1}{k^2} \sum_{k=1}^{N} |\alpha_k|^2 + |\beta_k|^2},
\]

and from this it follows that (on taking the limit as \(N \to \infty\))

\[
\sum_{k=1}^{\infty} \sqrt{|a_k|^2 + |b_k|^2} \leq \sqrt{\sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{k=1}^{\infty} |\alpha_k|^2 + |\beta_k|^2}
\]

which then shows that

\[
\sum_{k=1}^{\infty} \sqrt{|a_k|^2 + |b_k|^2}
\]

converges.

Having proved this, we use the fact that

\[ |a_k| \leq \sqrt{|a_k|^2 + |b_k|^2}, \quad |b_k| \leq \sqrt{|a_k|^2 + |b_k|^2} \]

in order to deduce that

\[
\sum_{k=1}^{\infty} (|a_k| + |b_k|) < +\infty.
\]

From this it follows that

\[ \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \]

converges uniformly on \([-\pi, \pi]\), by Weierstrass' Majorant Theorem because \(|a_k \cos kt + b_k \sin kt| \leq |a_k| + |b_k|\). This proves the result.