Lecture 2: Linear Algebra and Fourier Series.

1 Introduction.

At the beginning of the first lecture we gave the definition of Fourier series. Here we begin with the same definition:

**Definition 1.1** The Fourier series of a periodic function \( f(x) \) with period \( L \) is defined as the series

\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos \frac{2\pi k}{L} x + b_k \sin \frac{2\pi k}{L} x \right),
\]

where the coefficients \( a_k, b_k \) are defined as

\[
a_k = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos \frac{2\pi k}{L} x \, dx, \quad b_k = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin \frac{2\pi k}{L} x \, dx,
\]

with \( k = 0, 1, \ldots \) (note that this means that \( b_0 = 0 \)). This is called the real form of the Fourier series.

We also have the complex Fourier series of \( f(x) \):

**Definition 1.2** The complex Fourier series of a periodic function \( f(x) \) with period \( L \) is defined as the series

\[
\sum_{k=-\infty}^{\infty} c_k e^{i \frac{2\pi k}{L} x}
\]

where the coefficients \( c_k \) are defined by

\[
c_k = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-i \frac{2\pi k}{L} x} \, dx.
\]

**Note:** In this course we will always take

\[
L = 2\pi
\]

unless otherwise stated. Thus we have:
Definition 1.3  The Fourier series of a periodic function \( f(x) \) with period \( 2\pi \) is defined as the series

\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right),
\]
where the coefficients \( a_k, b_k \) are defined as

\[
a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx,
\]
with \( k = 0, 1, \ldots \) (note that this means that \( b_0 = 0 \)). This is called the real form of the Fourier series.

and

Definition 1.4  The complex Fourier series of a periodic function \( f(x) \) with period \( 2\pi \) is defined as the series

\[
\sum_{-\infty}^{\infty} c_k e^{ikx}
\]
where the coefficients \( c_k \) are defined by

\[
c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx.
\]

We are able to discuss the conditions of convergence of the series using the results we looked at in Lecture 1. In this lecture I want to place Fourier series in a mathematical framework which makes it easier to see the algebraic structure of the theory. This framework is provided by linear algebra.

2  Vector spaces with inner products.

We shall use the language of linear algebra to formulate and prove some results for Fourier series. First we recall the definition of inner product spaces.

Definition 2.1  Suppose \( V \) is a vector space over \( \mathbb{R} \) (the real numbers) or \( \mathbb{C} \) (the complex numbers). An inner product on \( V \) is a function

\[
\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} \quad \text{(if } V \text{ is a vector space over } \mathbb{R})
\]

or

\[
\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \quad \text{(if } V \text{ is a vector space over } \mathbb{C})
\]

with
1. \( \langle u, v \rangle \in \mathbb{R} \) for each \( u, v \in V \) (if \( V \) is a vector space over \( \mathbb{R} \))
2. \( \langle u, v \rangle \in \mathbb{C} \) for each \( u, v \in V \) (if \( V \) is a vector space over \( \mathbb{C} \))
3. \( \langle u, u \rangle \geq 0 \) for all \( u \in V \), and \( \langle u, u \rangle = 0 \) if and only if \( u = 0 \)
4. \( \langle au + bv, w \rangle = a \langle u, w \rangle + b \langle u, w \rangle \) for all \( u, v, w \in V \)
5. \( \overline{\langle u, v \rangle} = \langle v, u \rangle \) for all \( u, v \in V \)

From (4) and (5) it follows that
\[
\langle u, av + bw \rangle = \bar{a} \langle u, v \rangle + \bar{b} \langle u, w \rangle.
\]
Note that (5) reduces to \( \langle u, v \rangle = \langle v, u \rangle \) if \( V \) is a vector space over \( \mathbb{R} \).
Note that we have not made any assumption that the vector space has finite dimension.

This definition is familiar from linear algebra. For any inner product as defined above, we have the Cauchy-Schwarz-Bunyakovsky inequality:
\[
|\langle u, v \rangle| \leq \|u\| \cdot \|v\|
\]
where we use the inner product to define \( \|u\| \) as
\[
\|u\| = \sqrt{\langle u, u \rangle}.
\]
This defines a function
\[
\| \cdot \| : V \rightarrow \mathbb{R}_+
\]
and it is an example of a norm.

**Definition 2.2** By a norm on a vector space \( V \) we mean a function
\[
\| \cdot \| : V \rightarrow \mathbb{R}_+
\]
satisfying the following conditions:
1. \( \|u\| \geq 0 \) for all \( u \in V \) and \( \|u\| = 0 \) if and only if \( u = 0 \)
2. \( \|u + v\| \leq \|u\| + \|v\| \) for all \( u, v \in V \) (this is the familiar triangle inequality)
3. \( \|\lambda u\| = |\lambda| \|u\| \) for all \( u \in V \) and all \( \lambda \in \mathbb{C} \) if \( V \) is a complex vector space (all \( \lambda \in \mathbb{R} \) if \( V \) is a real vector space).

In this case, we say that \( V \) is a normed vector space (or a vector space with norm).

Before proceeding to look at some useful theorems, we mention some examples.
Example 2.1 $\mathbb{R}^n$ with inner product

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$$

with $x_k, y_k \in \mathbb{R}$.

Example 2.2 $\mathbb{C}^n$ with inner product

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k \bar{y}_k$$

with $x_k, y_k \in \mathbb{C}$.

Example 2.3 $C([a,b])$ = the set of all continuous, $\mathbb{C}$-valued functions on $[a,b]$ with inner product

$$\langle f, g \rangle = \int_{a}^{b} f(x)\overline{g(x)} dx.$$ 

It is a simple exercise to verify all the conditions required of an inner product in Example 2.3, except the requirement: $\langle f, f \rangle = 0$ only for $f = 0$. To prove this we have the following:

**Lemma 2.1** Suppose that $f(x)$ continuous and that

$$\int_{a}^{b} |f(x)|^2 dx = 0.$$ 

Then

$$f(x) = 0 \quad \text{for all} \quad x \in [a,b].$$

**Proof:** The condition

$$\int_{a}^{b} |f(t)|^2 dt = 0$$

then implies that

$$F'(x) = \int_{a}^{x} |f(t)|^2 dt = 0$$

for all $a \leq x \leq b$. Now, $f$, and therefore $|f(x)|^2$, is continuous, so $F(x)$ is differentiable and $F'(x) = |f(x)|^2 = 0$. 

4
Example 2.4 \(L^\infty([a,b])\), the set of all bounded, complex-valued functions on \([a,b]\). On this space we have the norm

\[ \|f\|_\infty = \sup_{x \in [a,b]} |f(x)|. \]

This is a norm which does not come from an inner product.

Example 2.5 \(L^1([a,b])\), the set of all complex-valued functions on \([a,b]\) satisfying

\[ \int_a^b |f(x)| \, dx < \infty. \]

On this space we have the norm

\[ \|f\|_1 = \int_a^b |f(x)| \, dx. \]

These are the so-called absolutely integrable functions on \([a,b]\). This is a norm which does not come from an inner product.

3 The space \(L^2([-\pi, \pi])\).

The natural setting for Fourier series is the space \(L^2([-\pi, \pi])\) which we define here:

Definition 3.1 \(L^2([-\pi, \pi])\) is the set of all complex-valued functions on \([-\pi, \pi]\) satisfying

\[ \int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty. \]

On this space we define the inner product:

\[ \langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} \, dx. \]

These are the so-called square-integrable functions on \([-\pi, \pi]\).

In particular we have the norm

\[ \|f\| = \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx}. \]

Before continuing, we look at some elementary examples of functions in \(L^2([-\pi, \pi])\).
We have
\[ \frac{1}{\sqrt{2}} \cos kt, \sin kt \in L^2([−\pi, \pi]) \]
for each \( k = 1,2,\ldots \). In fact
\[ \|\frac{1}{\sqrt{2}}\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} dt = 1 \]
\[ \|\cos kt\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 ktdt = 1 \quad \text{for } k = 1,2,\ldots \]
and
\[ \|\sin kt\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 ktdt = 1 \quad \text{for } k = 1,2,\ldots \]
\[ <\frac{1}{\sqrt{2}}, \cos kt> = 0, \quad <\frac{1}{\sqrt{2}}, \sin kt> = 0 \quad k = 1,2,\ldots \]
\[ <\cos mt, \sin nt> = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos mt \sin ntdt = 0 \]
as well as
\[ <\cos mt, \cos nt> = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos mt \cos ntdt = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \]
and
\[ <\sin mt, \sin nt> = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin mt \cos ntdt = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \]

These are easy to verify and they are left as exercises. These results say that the functions \( \cos kt, \sin kt \) are a collection of orthonormal vectors in \( L^2([−\pi, \pi]) \). We shall come back to this in the next section.

From the definition of the space \( L^2([−\pi, \pi]) \) we have the following elementary observation:

**Proposition 3.1** If \( f \in L^2([−\pi, \pi]) \) then its Fourier series is
\[ a_0 \frac{1}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \]
where
\[ a_k = <f, \cos kt> \quad (k = 0,1,2,\ldots), \quad b_k = <f, \sin kt> \quad (k = 1,2,\ldots) \]

**Proof:** The proof is by direct verification.
The question of convergence of this Fourier series is a natural one. One of the results we are going to prove is:

**Theorem 3.1** (Parseval’s Theorem) For \( f \in L^2([-\pi, \pi]) \) with Fourier series

\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),
\]

we have

\[
\|f\|^2 = \frac{|a_0|^2}{2} + \sum_{k=1}^{\infty} (|a_k|^2 + |b_k|^2).
\]

That is

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt = \frac{|a_0|^2}{2} + \sum_{k=1}^{\infty} (|a_k|^2 + |b_k|^2).
\]

Another result on convergence of the Fourier series is the following:

**Theorem 3.2** Let \( f \in L^2([-\pi, \pi]) \) and put

\[
S_N(t) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos kt + b_k \sin kt.
\]

Then \( S_N(t) \in L^2([-\pi, \pi]) \) and

\[
\|S_N - f\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |S_N(t) - f(t)|^2 dt \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.
\]

This is a new type of convergence and it is usually called **convergence in the mean**. Another name is **convergence in norm** or **strong convergence**. In fact we have the following definition:

**Definition 3.2** Let \( V \) be a vector space with norm \( \|\cdot\| \). Then we say that a sequence \( \{v_1, v_2, \ldots, v_n, \ldots\} \) of vectors in \( V \) **converges to** \( v \in V \) **in norm** (or **strongly**) if

\[
\|v_n - v\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
4 Orthogonal and orthonormal systems.

We have seen that the system of vectors in \( L^2([\pi, \pi]) \)

\[
\frac{1}{\sqrt{2}}, \cos kt, \sin kt \quad \text{with} \quad k = 1, 2, 3, \ldots
\]

is a set of orthonormal vectors. This motivates the following definition:

**Definition 4.1** An orthonormal system in a vector space \( V \) with inner product is a set of unit vectors \( u_1, u_2, u_3, \ldots \) (we may have infinitely many vectors) so that

\[
\langle u_k, u_l \rangle = 0 \quad \text{if} \quad k \neq l
\]

The importance of orthonormal systems is that they are, for infinite-dimensional vector spaces, the analogues of orthonormal bases (provided that there are enough vectors). However, the difference is that we may have an infinite number of vectors, and if we are to take a linear combination of them we must talk about the convergence of infinite sums of vectors. So, what do we mean by

\[
c_1e_1 + c_2e_2 + \cdots + c_ke_k + \cdots = \sum_{k=1}^{\infty} c_ke_k,
\]

where \( c_1, c_2, \ldots \) are constants and \( e_1, e_2, \ldots \) are vectors?

The answer to this question is provided by definition 3.2:

Defining the partial sum \( S_N \) as

\[
S_N = \sum_{k=1}^{N} c_ke_k, \quad N = 1, 2, 3, \ldots
\]

we define \( \sum_{k=1}^{\infty} c_ke_k \) as follows:

\[
S = \sum_{k=1}^{\infty} c_ke_k = \lim_{N \to \infty} S_N.
\]

With this definition, we can deal with convergence of an infinite sum of vectors (that is a series whose terms are vectors).

5 Generalised Pythagoras’ Theorem

Here we formulate a generalisation of Pythagoras theorem. The usual Pythagoras theorem says that, in a right-angled triangle, the sum of the squares of the two sides which are
orthogonal to each other is equal to the square of the hypotenuse. If one rewrites this in terms of vectors in two and three dimensions, we have the result that

$$||u + v||^2 = ||u||^2 + ||v||^2$$

provided that \(u \cdot v = 0\) (note that the sum of vectors \(u + v\) corresponds to the hypotenuse).

This well-known result can be generalised to vector spaces with inner products:

**Theorem 5.1** If \(u_1, u_2, \ldots, u_N\) is a set of orthogonal vectors in a vector space \(V\) with inner product, and if \(a_1, a_2, a_3, \ldots, a_N\) are complex numbers, then we have

$$\sum_{k=1}^{N} |a_k|^2 ||u_k||^2 = \sum_{k=1}^{N} |a_k|^2 ||u_k||^2.$$

**Proof:** A simple way of proving this is by induction. The theorem is obviously true for \(N = 1\). So assume we have proved it for \(N\). For \(N+1\) we have, on putting \(\sum_{k=1}^{N+1} a_k u_k = s_N\)

$$\sum_{k=1}^{N+1} |a_k|^2 ||u_k||^2 = \sum_{k=1}^{N} |a_k|^2 ||u_k||^2 + |a_{N+1}|^2 ||u_{N+1}||^2$$

by the induction hypothesis, where we have used the fact that \(u_{N+1}\) is orthogonal to all of \(u_1, \ldots, u_N\) to obtain \(\langle s_N, a_{N+1} u_{N+1} \rangle = 0\). Thus, by induction, the theorem is true for all \(N\).

As a corollary of this result we have the following:

**Theorem 5.2** For an orthonormal system \(\{e_1, \ldots, e_N\}\) in a vector space \(V\) with inner product, we have for any vector \(u \in \text{span}\{e_1, \ldots, e_N\}\) that

(i) \(u = \sum_{k=1}^{N} \langle u, e_k \rangle e_k\)

(ii) \(||u||^2 = \sum_{k=1}^{N} |\langle u, e_k \rangle|^2.\)
Proof: By definition we have
\[ u = a_1 e_1 + \cdots + a_N e_N \]
for some constants \( a_1, \ldots, a_N \). Then using the facts that (i) \( \langle e_i, e_j \rangle = 0 \) if \( i \neq j \) and (ii) \( \langle e_i, e_i \rangle = 1 \) for each \( i = 1, \ldots, N \), we calculate that \( \langle u, e_k \rangle = a_k \). This proves (1). Then part (2) follows from part (1) and Theorem 4.1.

Now, this result is true for arbitrary, finite \( N \), but it is not guaranteed to be true if we have an infinite number of linearly independent vectors in our vector space. However, we certainly have the following result for any orthonormal system in a vector space with inner product:

**Theorem 5.3 (Bessel’s inequality)** If \( \{e_1, e_2, \ldots, e_k, \ldots\} \) is an orthonormal system in a vector space \( V \) with inner product, then for any vector \( u \in V \)

\[
(i) \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2 \text{ converges;}
\]

\[
(ii) \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2 \leq \|u\|^2.
\]

For \( f \in L^2([-\pi, \pi]) \) with Fourier series

\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),
\]

Theorem 5.3 means that

\[
\frac{|a_0|^2}{2} + \sum_{k=1}^{\infty} |a_k|^2 + |b_k|^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt.
\]

**Proof of Theorem 5.3:** For each \( N \) we define

\[
S_N = \sum_{k=1}^{N} |\langle u, e_k \rangle|^2.
\]

Then define \( v_N = \langle u, e_1 \rangle e_1 + \cdots + \langle u, e_N \rangle e_N \): this is just the orthogonal projection of the vector \( u \) onto the subspace spanned by \( \{e_1, \ldots, e_N\} \). Therefore, \( u - v_N \) must be orthogonal to each of \( e_1, \ldots, e_N \). Further, \( u = (u - v_N) + v_N \) and it is trivial to check that \( u - v_N \perp v_N \) so that, by Pythagoras theorem, we have

\[
\|u\|^2 = \|u - v_N\|^2 + \|v_N\|^2
\]
from which it follows that
\[ \|v_N\|^2 \leq \|u\|^2. \]

Now, we also know (Theorem 4.2) that \( S_N = \|v_N\|^2 \), so we have shown that

\[ S_N \leq \|u\|^2 \]

for all \( N \). Since the terms in \( S_N \) are all non-negative, we have \( S_1 \leq S_2 \leq \cdots \leq S_N \leq \ldots \), that is \( \{S_N\} \) is an increasing sequence of real numbers, and it is bounded above by (by \( \|u\|^2 \)) so \( \lim_{N \to \infty} S_N \) exists and we have

\[ \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2 \leq \|u\|^2 \]

as claimed.

This result may seem rather disappointing in that we do not achieve the equality

\[ \|u\|^2 = \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2; \]

but we shall come back to this in the next theorem. However, as we have already seen, for \( f \in L^2([-\pi, \pi]) \) with Fourier series

\[ \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \]

we have

\[ \frac{|a_0|^2}{2} + \sum_{k=1}^{\infty} |a_k|^2 + |b_k|^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt. \]

so that the series \( \frac{|a_0|^2}{2} + \sum_{k=1}^{\infty} |a_k|^2 + |b_k|^2 \) is convergent, and therefore \( a_k \to 0, \ b_k \to 0 \) as \( k \to \infty \). In fact we have, as a corollary of Theorem 5.3, the **Riemann-Lebesgue Lemma**:

**Lemma 5.1 (Riemann-Lebesgue Lemma)**

\[ \langle u, e_k \rangle \to 0 \quad \text{as} \quad k \to \infty. \]

**Proof:**

\[ \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2 \]

converges, so \( |\langle u, e_k \rangle|^2 \to 0 \) as \( k \to \infty \) and hence \( \langle u, e_k \rangle \to 0 \) as \( k \to \infty \).

In particular, as we have noted:
Corollary 5.1  

(i) \( \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0 \)

(ii) \( \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0 \)

(iii) \( \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \sin(n + \frac{1}{2})x \, dx = 0 \)

Proof: The first two results have already been proved above. For the third part, notice that

\[
\sin(n + \frac{1}{2})x = \cos \frac{x}{2} \sin nx + \sin \frac{x}{2} \cos nx
\]

and then we have, with \( g(x) = f(x) \cos \frac{x}{2}, \ h(x) = f(x) \sin \frac{x}{2}, \) that

\[
\langle f, \sin(n + \frac{1}{2})x \rangle = \langle g, \sin nx \rangle + \langle h, \cos nx \rangle
\]

and then it follows immediately from the Riemann-Lebesgue Lemma that

\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \sin(n + \frac{1}{2})x \, dx = 0.
\]

6  Closed systems and expansions in terms of orthonormal sets.

We now come to an important result: how to represent vectors in terms of a given orthonormal system of vectors. In order to formulate it we need some more mathematical language.

First, we agree to write

\[
u = \sum_{k=1}^{\infty} a_k v_k
\]

where \( v_1, v_2, \ldots \) is a sequence of vectors in a normed vector space \( V, \) if the sequence

\[
u_N = \sum_{k=1}^{N} a_k v_k
\]

converges to \( u: \) that is, if

\[\|u - u_N\| \to 0 \text{ as } N \to \infty \]

in \( V.\)
Definition 6.1 If $V$ is a vector space with inner product and \{e_1, \ldots, e_k, \ldots\} is an (infinite) orthonormal system in $V$, then we say that \{e_1, \ldots, e_k, \ldots\} is a **closed system** if, for each $u \in V$ we have

$$u = \sum_{k=1}^{\infty} \langle u, e_k \rangle e_k.$$  

That is, if the sequence \{u_N\} defined by

$$u_N = \sum_{k=1}^{N} \langle u, e_k \rangle e_k$$

converges to $u$:

$$\|u - u_N\| \to 0 \text{ as } N \to \infty.$$  

Then we have the following important theorem:

**Theorem 6.1** The orthonormal system \{e_1, \ldots, e_k, \ldots\} in a vector space $V$ with inner product is closed if and only if

$$\|u\|^2 = \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2$$

for each $u \in V$. That is, if and only if equality is obtained in Bessel’s inequality.

Proof: As we have seen above, $u = (u - u_N) + u_N$ and $u_N \perp (u - u_N)$ so that

$$\|u\|^2 = \|u_N\|^2 + \|u - u_N\|^2$$

by Pythagoras’ theorem. Then we have

$$\|u_N\|^2 = \sum_{k=1}^{N} |\langle u, e_k \rangle|^2$$

and, by Bessel’s inequality, we have

$$\lim_{N \to \infty} \|u_N\|^2 = \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2$$

exists. Thus, if the system is closed we have $\|u - u_N\| \to 0$ as $N \to \infty$ and hence

$$\lim_{N \to \infty} \|u_N\|^2 = \|u\|^2,$$

so that
\[ \|u\|^2 = \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2. \]

On the other hand, if we know that
\[ \|u\|^2 = \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2. \]
then
\[ \|u\|^2 - \|u_N\|^2 \to 0 \text{ as } N \to \infty, \]
and hence we must have
\[ \|u - u_N\| \to 0 \text{ as } N \to \infty, \]
and this shows that the system is closed.

What does all of this mean? We shall look at two examples which are of central importance for us.

**Theorem 6.2** The orthonormal system
\[
\frac{1}{\sqrt{2}}, \cos kt, \sin kt, \quad k = 0, 1, 2, \ldots
\]
is a closed system in \(L^2([-\pi, \pi])\) equipped with the inner product
\[
\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt.
\]
In particular we have
\[
f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt),
\]
where equality is meant as being between vectors in \(L^2([-\pi, \pi])\). This means that
\[
\|f - S_N\| \to 0 \text{ as } N \to \infty,
\]
where \(S_N = \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kt + b_k \sin kt)\).

In order to avoid the appearance of putting \(f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)\) at each \(t\) and in order to emphasize the fact that we have equality of two vectors in \(L^2([-\pi, \pi])\) we write
\( f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt). \)

**Theorem 6.3** Take the same space \( L^2([-\pi, \pi]) \) and use the inner product

\[ \langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)g(t)dt. \]

Then \( \{e^{ikt} : k = 0, \pm 1, \pm 2, \ldots \} \) is a closed orthonormal system. In particular

\[ f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \]

where \( c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ikt}dt. \) Again, equality here is meant as vectors in \( L^2([-\pi, \pi]). \)

### 7 Parseval’s Theorem

Now we come to Parseval’s Theorem:

**Theorem 7.1** *(Parseval’s Theorem)* Let \( V \) be a vector space with inner product, and let \( \{e_1, e_2, \ldots \} \) be a closed orthonormal system in \( V \). Then we have for any \( u, v \in V \)

\[ \langle u, v \rangle = \sum_{k=1}^{\infty} a_k \overline{b_k} \]

where \( a_k = \langle u, e_k \rangle, \ b_k = \langle v, e_k \rangle. \) In particular

\[ \|u\|^2 = \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2 = \sum_{k=1}^{\infty} |a_k|^2. \]

**Proof:** Note that we already have, for a closed system \( \{e_1, e_2, \ldots \} \), that

\[ \|u\|^2 = \sum_{k=1}^{\infty} |a_k|^2, \quad \|v\|^2 = \sum_{k=1}^{\infty} |b_k|^2 \]

(on using our definitions for the notation). Then note that we have

\[ \|u - v\|^2 = \langle u - v, u - v \rangle \]
\[ = \langle u, u \rangle + \langle u, v \rangle - 2\langle u, v \rangle \]
\[ = \|u\|^2 + \|v\|^2 - 2\langle u, v \rangle, \]

from which we deduce that
\[2\langle u, v \rangle = \|u - v\|^2 - \|u\|^2 - \|v\|^2.\]

Then note that

\[
\|u - v\|^2 = \sum_{k=1}^{\infty} |a_k - b_k|^2
= \sum_{k=1}^{\infty} |a_k|^2 + \sum_{k=1}^{\infty} |b_k|^2 - 2 \sum_{k=1}^{\infty} a_k \bar{b}_k
= \|u\|^2 + \|v\|^2 - 2 \sum_{k=1}^{\infty} a_k \bar{b}_k,
\]

as a straightforward calculation shows. Comparing the two sets of equations, we obtain

\[\langle u, v \rangle = \sum_{k=1}^{\infty} a_k \bar{b}_k.\]

This proves the result.

Now for some examples.

**Example 7.1** We look at \(L^2([-\pi, \pi])\) with the inner product

\[\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)g(t)dt.\]

Then the complex Fourier series of a periodic function \(f(t) \in L^2([-\pi, \pi])\) is, as already noted,

\[\sum_{n=-\infty}^{\infty} c_n e^{int}\]

where

\[c_n = \langle f, e^{int} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt.\]

Then we find from Parseval’s Theorem that

\[\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2,
\]

or

\[\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2.\]
Example 7.2 Take the same space $L^2([-\pi, \pi])$ and use the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)dt.$$  

The Fourier series of a periodic function $f(t) \in L^2([-\pi, \pi])$ is

$$a_0 = \frac{1}{2} \left( a_k \cos kx + b_k \sin kx \right),$$

where the coefficients $a_k, b_k$ are defined as

$$a_k = \langle f(x), \cos kx \rangle, \quad b_k = \langle f(x), \sin kx \rangle, \quad k = 0, 1, 2, \ldots$$

Using Parseval’s Theorem we find

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left( |a_k|^2 + |b_k|^2 \right).$$

As a final example, we look at a particular function.

Example 7.3 Let $f(t)$ be a periodic function of period $2\pi$ and $f(t) = t, \quad -\pi < t \leq \pi$. We have $f \in L^2([-\pi, \pi])$, as is easily verified. The Fourier series of $f(t)$ is (I leave the calculations to you)

$$f \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kt$$

since we find that $a_k = 0, \quad k = 0, 1, \ldots$ whereas $b_k = \frac{2(-1)^{k+1}}{k}$. Then Parseval’s theorem give us

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt = \sum_{k=1}^{\infty} |b_k|^2 = \sum_{k=1}^{\infty} \frac{4}{k^2},$$

and, calculating the left-hand side we find

$$\sum_{k=1}^{\infty} \frac{4}{k^2} = \frac{2\pi^2}{3}$$

from which we see that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$  

Now you have the answer to what you have been wondering about since Analysis B: what is the value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$, and how do you calculate it?