Lecture 1.
Functional series.
Pointwise and uniform convergence.

1 Introduction.

In this course we study amongst other things Fourier series. The Fourier series for a periodic function \( f(x) \) with period \( 2\pi \) is defined as the series

\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),
\]

where the coefficients \( a_k, b_k \) are defined as

\[
a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx,
\]

with \( k = 0, 1, \ldots \) (note that this means that \( b_0 = 0 \)).

This is an example of a functional series, which is a series whose terms are functions:

\[
\sum_{k=0}^{\infty} u_k(x).
\]

As usual with series, we define the above infinite sum as a limit:

\[
\sum_{k=0}^{\infty} u_k(x) = \lim_{N \to \infty} \sum_{k=0}^{N} u_k(x),
\]

providing the limit exists. Note that different values of \( x \) will, in general, give different limits, if they exist.

In this lecture we shall look at functional series, and functional sequences, and we shall consider first the question of convergence. To deal with this, we consider two types of convergence: pointwise convergence and uniform convergence. There are three main results: the first one is that uniform convergence of a sequence of continuous functions gives us a continuous function as a limit. The second main result is Weierstrass’ Majorant Theorem, which gives a condition that guarantees that a functional series converges to a continuous function. The third result is that integrals of a sequence of functions which converges uniformly to a limit function \( f(x) \) also converge with the limit being the integral of \( f(x) \). These results are not only good for your mental health, they are also important tools in our later discussion of Fourier series, and that is the reason for looking at them.
2 Power series.

A power series in the variable $x$ is a series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{k=0}^{\infty} a_kx^k,$$

where the coefficients $a_0, a_1, a_2, \ldots$ are real or complex numbers.

We are (in principle) allowed to put the variable $x$ equal to any number we wish. For instance, with the power series

$$\sum_{k=0}^{\infty} a_kx^k$$

we may put $x = 2$ and we obtain the numerical series

$$\sum_{k=0}^{\infty} a_k2^k = a_0 + 2a_1 + 4a_2 + 8a_3 + \cdots$$

Or, with the power series

$$\sum_{k=0}^{\infty} x^k$$

(note that in this case all the coefficients are equal to 1) we obtain

$$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + 8 + 16 + \cdots$$

if we put $x = 2$.

However, we have a problem: how do we know that the numerical series we obtain by putting $x = 2$ in the power series

$$\sum_{k=0}^{\infty} a_kx^k$$

is convergent?

We can always put $x = 2$ and then investigate the convergence of the numerical series, but this is a rather inefficient way of deciding if a particular value of $x$ gives us a convergent series. So we come to the following question:

Given the power series

$$\sum_{k=0}^{\infty} a_kx^k,$$

with a given choice of coefficients $a_0, a_1, a_2, \ldots$, what values of $x$ give us convergent series, and which values give divergent series?

There are two very useful results which help us in examining this question. The first one is the following:
Theorem 2.1 For a power series
\[ \sum_{k=0}^{\infty} a_k x^k \]
there are three possibilities:

1. The power series \( \sum_{k=0}^{\infty} a_k x^k \) diverges for all \( x \neq 0 \)

2. The power series \( \sum_{k=0}^{\infty} a_k x^k \) converges for all values of \( x \)

3. There is a positive number \( R \) such that \( \sum_{k=0}^{\infty} a_k x^k \) converges for all values of \( x \) with \( |x| < R \) and diverges for all values of \( x \) with \( |x| > R \).

At first sight, this looks like a very useless result, because it doesn’t answer the question of which values of \( x \) are allowed. However, it is a very useful result: it tells us what sort of behaviour we can expect, and what to look for in a power series. In particular, it tells us what is the decisive factor in our subsequent investigation: we need to find the number \( R \), which is called the radius of convergence of the power series. One word of warning: the theorem tells us that, when we have found the radius of convergence \( R \), then the series converges for \( |x| < R \) and diverges whenever \( |x| > R \), but it doesn’t say anything about the case when \( |x| = R \), that is, when \( x = R \) and \( x = -R \). The theorem is silent on this matter. In fact, we must investigate the cases \( x = \pm R \) separately (that is, we put in \( x = R \) or \( x = -R \) and we investigate the convergence of the series that arise). This is a small price to pay when we know what \( R \) is.

So, given our theorem, how do we go about calculating \( R \)? One result is the following:

Theorem 2.2 Given the power series \( \sum_{k=0}^{\infty} a_k x^k \), suppose that one of the following limits exist:
\[ K = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|, \quad K = \lim_{k \to \infty} k \sqrt[1]{|a_k|}. \]
Then the following is true:

1. If \( K = 0 \) then the power series \( \sum_{k=0}^{\infty} a_k x^k \) converges for all values of \( x \);

2. If \( K > 0 \), then the radius of convergence \( R \) of the power series \( \sum_{k=0}^{\infty} a_k x^k \) is
\[ R = \frac{1}{K}. \]
If either of the limits

\[ \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|, \quad \lim_{k \to \infty} k^{\sqrt{|a_k|}}. \]

fails to exist, then the power series \( \sum_{k=0}^{\infty} a_k x^k \) diverges for all values of \( x \neq 0 \).

This theorem is proved using the following result (the proof is given in Mats Neymark’s *Kompendium om konvergens*):

**Theorem 2.3** Suppose \( \sum_{k=0}^{\infty} a_k \) is a numerical series and suppose that one of the following limits exists

\[ K = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|, \quad K = \lim_{k \to \infty} k^{\sqrt{|a_k|}}. \]

If \( 0 \leq K < 1 \) then the series converges absolutely. If \( K > 1 \) then the series diverges. If \( K = 1 \) then convergence or divergence of the series must be investigated using some other method.

### 3 Pointwise Convergence.

Consider the power series

\[ \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{for } |x| < 1. \]

This just says that for each \( x \in ]-1,1[ \) the power series in the left-hand side converges to the number \( 1/(1-x) \). If we put

\[ f_n(x) = \sum_{k=0}^{n} x^k, \quad f(x) = \frac{1}{1-x}, \]

then we can rephrase this as

\[ f_n(x) \to f(x) \quad \text{as } n \to \infty \quad \text{for each } x \in ]-1,1[. \]

We give this type of convergence a name: **pointwise convergence**. Note that we have first defined a sequence of functions \( f_n \) by putting

\[ f_n(x) = \sum_{k=0}^{n} x^k \]

for each \( n = 0, 1, 2, \ldots \).
Definition 3.1 (Pointwise convergence.) Suppose \( \{ f_n(x) : n = 0, 1, 2, \ldots \} \) is a sequence of functions defined on an interval \( I \). We say that \( f_n(x) \) converges pointwise to the function \( f(x) \) on the interval \( I \) if

\[
f_n(x) \to f(x), \quad \text{as} \quad n \to \infty, \quad \text{for each} \quad x \in I.
\]

We call the function \( f(x) \) the limit function.

Example 3.1 \( f_n(x) = x - \frac{1}{n} \). Then \( f_n(x) \) converges pointwise to \( x \) for each \( x \in \mathbb{R} \):

\[
|f_n(x) - x| = \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty.
\]

Example 3.2 \( f_n(x) = e^{-nx} \) on \([1, 3]\). For each \( x \in [1, 3] \) we have \( nx \to \infty \) as \( n \to \infty \) and therefore \( f_n(x) \to 0 \) as \( n \to \infty \) for each \( x \in [1, 3] \). Thus \( f_n(x) \) converges pointwise to \( f(x) = 0 \) for each \( x \in [1, 3] \).

Example 3.3 \( f_n(x) = e^{-nx} \) on \([0, 3]\). For each \( 0 < x \leq 3 \) we have \( nx \to \infty \) as \( n \to \infty \) and therefore \( f_n(x) \to 0 \) as \( n \to \infty \) for each \( 0 < x \leq 3 \). However, at \( x = 0 \) we have \( f_n(0) = 1 \) for all \( n \). Thus \( f_n(x) \) converges pointwise to the function \( f(x) \) defined by \( f(0) = 1, f(x) = 0 \) for each \( 0 < x \leq 3 \). This is not a continuous function, despite the fact that each function \( f_n(x) \) is continuous.

The last example shows what can happen with pointwise convergence: the limit function may fail to be continuous, even though all functions in the sequence are continuous.

Example 3.4 Let the sequence \( f_n \) be defined as

\[
f_n(x) = \frac{n^2 x}{(nx + 1)^3} \quad x \in [0, \infty[.
\]

Then \( f_n(0) = 0 \) and for each fixed \( x > 0 \)

\[
f_n(x) = \frac{n^2 x}{(nx + 1)^3} = \frac{n^2 x}{n^3(x + \frac{1}{n})^3} = \frac{1}{n} \frac{x}{(x + \frac{1}{n})^3} \to 0 \quad \text{as} \quad n \to \infty.
\]

So that \( f_n(x) \to f(x) = 0 \) pointwise on \([0, \infty[\).

Then for \( x \geq 0 \) we have

\[
f_n'(x) = \frac{n^2(1 - 2nx)}{(nx + 1)^4}
\]

and we see that for \( x > 0 \) we have \( f_n'(x) \to 0 \) as \( n \to \infty \) whereas \( f_n'(0) = n^2 \to \infty \). Here we see that \( f_n' \to f' \) only on for \( x > 0 \). This shows that differentiability is not always respected by pointwise convergence.
The last two examples then lead us to pose the question: what extra condition (other than just pointwise convergence) can guarantee that the limit function is also continuous or differentiable? The answer to this is given by the concept of uniform convergence.

4 Uniform convergence

We define for a real-valued (or complex-valued) function $f$ on a non-empty set $I$ the supremum norm of $f$ on the set $I$:

$$\|f\|_I = \sup_{x \in I} |f(x)|.$$  

Note that if $f$ is a bounded function on $I$ then

$$\sup_{x \in I} |f(x)| = \sup\{|f(x)| : x \in I\}$$  

exists, by the so-called supremum axiom. Observe that

$$|f(x)| \leq \|f\|_I \quad \text{for all } x \in I,$$

and that $|f(x)|$ takes on values which are arbitrarily near $\|f\|_I$. In particular $\|f\|_I$ is the largest value of $|f(x)|$ whenever such a value exists (such as when $I$ is a closed, bounded interval and $f(x)$ is a continuous function on $I$).

The supremum norm has the following properties for functions $f$ and $g$ on a set $I$:

$$\|f\|_I \geq 0 \quad \text{and} \quad \|f\|_I = 0 \iff f(x) = 0 \quad \text{for all } x \in I$$  

$$\|cf\| = |c| \cdot \|f\|_I \quad \text{for any constant } c$$  

$$\|f + g\|_I \leq \|f\|_I + \|g\|_I \quad \text{(triangle inequality)}$$  

$$\|f\|_J \leq \|f\|_I \quad \text{when } J \text{ is a subset of } I.$$  

The proof of these properties is left as an exercise for the interested reader.

Now we come to the definition of uniform convergence:

**Definition 4.1** A sequence of functions $f_n(x)$ defined on an set $I$ is said to converge uniformly to $f(x)$ on $I$ if

$$\|f_n - f\|_I \to 0 \quad \text{as } n \to \infty.$$  

We write this as

$$\lim_{n \to \infty} f_n = f \quad \text{uniformly on } I$$  

or as
Uniform convergence implies pointwise convergence, however there are sequences which converge pointwise but not uniformly. Indeed we have

\[ |f_n(x) - f(x)| \leq \sup_{x \in I} |f_n(x) - f(x)| = \|f_n - f\|_I, \]

so that

\[ f_n \to f \text{ uniformly on } I \text{ as } n \to \infty \]

\[ \implies |f_n(x) - f(x)| \to 0 \text{ for each } x \in I \]

\[ \implies f_n \to f \text{ pointwise on } I. \]

We record this as a result:

**Lemma 4.1** If the sequence of functions \( f_n(x) \) converges uniformly to \( f(x) \) on the interval \( I \), then \( f_n(x) \) converges pointwise to \( f(x) \).

This Lemma says that the limit function obtained through uniform convergence (if this occurs) is the same as the limit function obtained from pointwise convergence. Or: if \( f_n(x) \) converges to \( f(x) \) uniformly, then it must converge to \( f(x) \) pointwise. This then tells us how to go about testing for uniform convergence: first, obtain the pointwise limit \( f(x) \) and then see if we have uniform convergence to \( f(x) \).

**Example 4.1** \( f_n(x) = e^{-nx} \) on \([1, 3]\). We have seen above that \( f_n(x) \) converges pointwise to \( f(x) = 0 \) for each \( x \in [1, 3] \). Then we have \( |f_n(x) - f(x)| = |f_n(x)| \) and we then have

\[ \|f_n - f\| = \sup_{x \in [1,3]} |f_n(x)| \]

\[ = \sup_{x \in [1,3]} e^{-nx} \]

\[ = \sup_{x \in [1,3]} e^{-nx} \]

\[ = e^{-n} \to 0 \text{ as } n \to \infty. \]

Thus we have uniform convergence in this case. Note that the last step follows from the observation that \( e^{-nx} \) is strictly decreasing for \( x \geq 0 \) with \( n \geq 0 \), so that \( e^{-n} \geq e^{-nx} \) for all \( x \geq 1 \).
Example 4.2 \( f_n(x) = xe^{-nx} \) on \( I = [0, \infty[ \). Here the interval is unbounded. First we look at pointwise convergence: \( f_n(0) = 0 \) and for \( x > 0 \) we have that \( f_n(x) \to 0 \) as \( n \to \infty \). Thus \( f_n(x) \to 0 \) pointwise on \( I \). We now need to investigate uniform convergence. Since the limit function \( f(x) = 0 \) we have

\[
\|f_n - f\| = \sup_{x \in [0, \infty[} |xe^{-nx}|
\]

because \( f_n(x) \geq 0 \) for \( x \geq 0 \). Now, we have \( f'_n(x) = (1 - nx)e^{-nx} \) for \( x > 0 \) (observe that you should never differentiate on closed intervals), and we see that \( f'_n(x) = 0 \) when \( x = 1/n \). Further, \( f'_n(x) > 0 \) for \( 0 < x < 1/n \), and \( f'_n(x) < 0 \) for \( x > 1/n \), so we conclude that \( f_n(x) \) has a maximum at \( x = 1/n \) and hence

\[
\|f_n - f\| = \sup_{x \in [0, \infty[} xe^{-nx}
\]

\[
= f_n\left(\frac{1}{n}\right)
\]

\[
= \frac{1}{ne} \to 0 \quad \text{as} \quad n \to \infty.
\]

So we see that the sequence of functions \( f_n(x) = xe^{-nx} \) converges uniformly to 0 on the interval \( I = [0, \infty[ \).

5 Uniform convergence and continuity.

We now come to two important results. The first is the following.

Theorem 5.1 Suppose \( f_n(x) \) is a sequence of continuous functions on an interval \( I \) and suppose also that \( f_n(x) \) converges uniformly to \( f(x) \) on the interval \( I \). Then the limit function \( f(x) \) is also continuous.

Proof: First note that for \( x, a \in I \) we may write

\[
f(x) - f(a) = [f(x) - f_n(x)] + [f_n(x) - f_n(a)] + [f_n(a) - f(a)]
\]

from which we obtain (using the triangle inequality)

\[
|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|
\]

Then note that, because \( \sup_{x \in I} |f(x) - f_n(x)| \to 0 \) as \( n \to \infty \) we have for any given choice of \( \epsilon > 0 \) a natural number \( N \) such that

\[
\sup_{x \in I} |f(x) - f_n(x)| < \frac{\epsilon}{3}
\]
for all \( n \geq N \). We also have

\[
|f(a) - f_n(a)| < \frac{\epsilon}{3}
\]

since

\[
|f(a) - f_n(a)| \leq \sup_{x \in I} |f(x) - f_n(x)| < \frac{\epsilon}{3}.
\]

Using this in equation (5.1), we have, for a given \( \epsilon > 0 \), a natural number \( N \) so that

\[
|f(x) - f(a)| \leq \frac{2\epsilon}{3} + |f_n(x) - f_n(a)|
\]

for \( n \geq N \). Fix the choice of \( n \geq N \), say \( n = N \). We now use continuity of the \( f_n(x) \): for each \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
|f_n(x) - f_n(a)| < \frac{\epsilon}{3} \quad \text{whenever} \quad |x - a| < \delta.
\]

Consequently, for \( |x - a| < \delta \) we have

\[
|f(x) - f(a)| \leq \frac{2\epsilon}{3} + |f_n(x) - f_n(a)| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

This means that for any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
|f(x) - f(a)| < \epsilon \quad \text{whenever} \quad |x - a| < \delta
\]

and this means that \( f(x) \to f(a) \) as \( x \to a \). That is, \( f(x) \) is continuous at each \( a \in I \).

**Remark:** This result is very useful as a quick test for the absence of uniform convergence: if

(i) \( f_n(x) \), \( n = 0, 1, 2, \ldots \) is a sequence of continuous functions (on some interval);

(ii) \( f_n(x) \) converges pointwise to \( f(x) \);

(iii) \( f(x) \) is not continuous;

(iv) Then \( f_n(x) \) does not converge uniformly to \( f(x) \).

**Example 5.1** \( f_n(x) = e^{-nx} \) on \([0, 3]\) is a sequence of continuous functions, converging pointwise to \( f(x) \) defined by

\[
f(x) = \begin{cases} 
1 & \text{for } x = 0 \\
0 & \text{for } 0 < x \leq 3,
\end{cases}
\]

which is not continuous, and so, by Theorem 4.1, the sequence does not converge uniformly to \( f(x) \).

The second result we mention is the following, which is of great use in integrating series:
Theorem 5.2 Suppose that \( f_n(x) \) is sequence of continuous functions which converges uniformly to a continuous function \( f(x) \) on a bounded interval \([a, b]\). Then we have

\[
\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b \lim_{n \to \infty} f_n(x) \, dx = \int_a^b f(x) \, dx.
\]

**Proof:** The proof is quite simple:

\[
\left| \int_a^b f_n(x) \, dx - \int_a^b f(x) \, dx \right| = \left| \int_a^b (f_n(x) - f(x)) \, dx \right|
\leq \int_a^b |f_n(x) - f(x)| \, dx
\leq \int_a^b \|f_n - f\| \, dx
= \|f_n - f\| \int_a^b 1 \, dx
= \|f_n - f\| (b - a) \to 0 \text{ as } n \to \infty.
\]

Thus:

\[
\int_a^b f_n(x) \, dx \to \int_a^b f(x) \, dx
\]
as \( n \to \infty \) if \( f_n \to f \) uniformly on \( I \), which is what we wanted to prove.

**Remark 5.1** Theorem 5.2 is proved here for continuous functions so that the integrals exist. It is however possible to replace the word continuous by the word integrable, and the theorem is still true.

We now give a result about uniform convergence and differentiability: it tells us under which conditions the limit function \( f(x) \) is differentiable whenever the functions of the sequence \( f_n(x) \) are differentiable.

**Theorem 5.3** Suppose that \( \{f_n(x); n = 0, 1, 2, \ldots\} \) is a sequence of functions on an interval \( I \) and satisfying the following conditions:

(i) \( f_n(x) \) is differentiable on \( I \) for each \( n = 0, 1, 2, \ldots \)

(ii) \( f_n(x) \) converges pointwise to \( f(x) \) on \( I \)

(iii) \( f'_n(x) \) is continuous for each \( n \) and \( f'_n \to g \) converges uniformly on \( I \) where \( g(x) \) is a continuous function on \( I \).

Then the limit function \( f(x) \) is differentiable and \( f'(x) = g(x) \).
This result is very useful, as we shall see, in examining the differentiability of functional series.

**Proof:** Because of differentiability we have (for \( a, x \in I \))

\[
 f_n(x) = f_n(a) + \int_a^x f'_n(t) \, dt
\]

Furthermore, since \( f'_n(t) \to g(t) \) uniformly on \( I \), we know from Theorem 5.2 that

\[
 \int_a^x f'_n(t) \, dt \to \int_a^x g(t) \, dt
\]

when \( n \to \infty \). Also, \( f_n(x) \to f(x) \) pointwise as \( n \to \infty \). From this it follows that

\[
 f(x) = f(a) + \int_a^x g(t) \, dt
\]

on letting \( n \to \infty \). Since \( g(x) \) is continuous (it is the uniform limit of a sequence of continuous functions, so it is continuous by Theorem 5.1), the integral exists and is a primitive function of \( g(x) \). Differentiating this last equation, we obtain

\[
 f'(x) = g(x).
\]

This concludes the proof.

6 Applications to functional series.

**Definition 6.1** A functional series is a series

\[
 \sum_{k=0}^{\infty} u_k(x)
\]

where each term of the series \( u_k(x) \) is a function on an interval \( I \).

We can also define **pointwise convergence** for functional series:

**Definition 6.2** The functional series

\[
 \sum_{k=0}^{\infty} u_k(x)
\]

is pointwise convergent for each \( x \in I \) if the limit

\[
 \sum_{k=0}^{\infty} u_k(x) = \lim_{N \to \infty} \sum_{k=0}^{N} u_k(x)
\]

exists for each \( x \in I \).
Thus, we always define a **sequence of partial sums** $S_N(x)$ given as

$$S_N(x) = \sum_{k=0}^{N} u_k(x)$$

so that

$$S_0(x) = u_0(x), \quad S_1(x) = u_0(x) + u_1(x), \quad S_2(x) = u_0(x) + u_1(x) + u_2(x), \ldots$$

and if

$$\lim_{N \to \infty} S_N(x)$$

exists for $x$ then we say that the series

$$\sum_{k=0}^{\infty} u_k(x) = \lim_{N \to \infty} S_N(x)$$

converges at $x$. It converges pointwise on the interval $I$ if

$$\lim_{N \to \infty} S_N(x)$$

exists for each $x \in I$.

With these definitions, we deduce from Theorem 4.1 that if the functions $u_k(x)$ are all continuous on $I$ and if the sequence of partial sums $S_N(x)$ converges uniformly to $S(x)$ on $I$, then $S(x)$ is continuous. However, we would like an **efficient way** of deciding if a functional series converges uniformly to a (unique) limit. It is not at all easy to apply the definition of uniform convergence to an infinite sum of functions, so another method is desirable. The appropriate result is **Weierstrass’ Majorant Theorem:**

**Theorem 6.1** Suppose that the functional series

$$\sum_{k=0}^{\infty} u_k(x)$$

is defined on an interval $I$ and that there is a **sequence of positive constants** $M_k$ so that

$$|u_k(x)| \leq M_k, \quad k = 0, 1, 2, \ldots$$

for all $x \in I$. If

$$\sum_{k=0}^{\infty} M_k$$

converges, then

$$\sum_{k=0}^{\infty} u_k(x)$$

converges uniformly on $I$. 

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Proof: If the conditions are fulfilled then we immediately have, from the Comparison Theorems for Positive Series, that, for each \( x \in I \), the series
\[
\sum_{k=0}^{\infty} |u_k(x)|
\]
is convergent, so that
\[
\sum_{k=0}^{\infty} u_k(x)
\]
is absolutely convergent, and therefore convergent. This means that
\[
\sum_{k=0}^{\infty} u_k(x)
\]
is pointwise convergent on \( I \), and we denote the limit by \( S(x) \). We now show that the partial sums
\[
S_N(x) = \sum_{k=0}^{N} u_k(x)
\]
converges uniformly to \( S(x) \) on \( I \) under the conditions of the theorem. We have
\[
S(x) - S_N(x) = \sum_{k=N+1}^{\infty} u_k(x)
\]
(all we do is subtract the first \( N \) terms from the series). Then it follows that
\[
|S(x) - S_N(x)| \leq \sum_{k=N+1}^{\infty} |u_k(x)| \leq \sum_{k=N+1}^{\infty} M_k
\]
for each \( x \in I \), since \( |u_k(x)| \leq M_k \) for each \( x \in I \) according to our assumption. Then
\[
\|S - S_N\|_I \leq \sum_{k=N+1}^{\infty} M_k.
\]
We also know (by assumption) that \( \sum_{k=0}^{\infty} M_k \) converges, so we must have that \( \sum_{k=N+1}^{\infty} M_k \to 0 \) as \( N \to \infty \). Consequently,
\[
\|S - S_N\|_I \to 0 \quad \text{as} \quad N \to \infty,
\]
and our result is proved.

Corollary 6.1 If

(i) the functional series

\[
S(x) = \sum_{k=0}^{\infty} u_k(x) \quad \text{converges uniformly on interval} \ I,
\]
(ii) \( u_k(x) \) is a continuous function on \( I \) for each \( k = 0, 1, 2, \ldots \),
then \( S(x) \) is continuous on \( I \).

**Proof:** Because a finite sum of continuous functions is again a continuous function, it follows that the partial sums

\[
S_N(x) = \sum_{k=0}^{N} u_k(x)
\]

are continuous functions for \( N = 0, 1, 2, \ldots \). Then by Theorem 5.1, we have that \( S(x) = \lim_{N \to \infty} S_N(x) \) is a continuous function.

**Example 6.1** Take the functional series

\[
\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}.
\]

We have

\[
|u_k(x)| = \left| \frac{\sin kx}{k^2} \right| = \left| \frac{\sin kx}{k^2} \right| \leq \frac{1}{k^2}
\]

since \( |\sin t| \leq 1 \) for all real \( t \). We know (standard positive series) that

\[
\sum_{k=1}^{\infty} \frac{1}{k^2}
\]

converges (series of the form \( \sum 1/k^\alpha \) converge for \( \alpha > 1 \) and diverge for \( \alpha \leq 1 \)). Hence, by Weierstrass' Majorant Theorem,

\[
\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}
\]

converges uniformly for all \( x \), and by Corollary 5.1 this series is a continuous function of \( x \) for all \( x \in \mathbb{R} \).

**Remark 6.1** One advantage of Weierstrass' Majorant Theorem is that we do not have to calculate the value of the series at each \( x \in I \) in order to decide if we have uniform convergence. However, a drawback is that the conditions of the theorem are only sufficient to establish uniform convergence, they are not absolutely necessary for uniform convergence. In the final section of these lecture notes we give a necessary and sufficient condition for uniform convergence.

**Remark 6.2** In our statement of Weierstrass’ Majorant Theorem, we have not said anything about how to find the constants \( M_k \). Usually we take

\[
M_k = \sup_{x \in I} |u_k(x)|,
\]

but this is not strictly necessary: any sequence (of constants) will do provided that \( \sum M_k \) converges.
Another result of interest is the following:

**Theorem 6.2** If

(i) the functional series

\[
\sum_{k=0}^{\infty} u_k(x)
\]

converges uniformly on the interval \( I \)

(ii) \( u_k(x) \) is continuous on \( I \) for each \( k = 0, 1, 2, \ldots \),

then

\[
\int_a^x \left( \sum_{k=0}^{\infty} u_k(t) \right) \, dt = \sum_{k=0}^{\infty} \left( \int_a^x u_k(t) \, dt \right)
\]

for all \( a, x \in I \). In other words, if the series of continuous functions converges uniformly on \( I \), then the integral of the sum is the sum of the integrals of the functions, just as in the case of a finite sum.

**Proof:** See *Kompendium om Konvergens*.

We can also say something about the differentiability of the series \( \sum u_k(x) \), using Theorem 5.3. In this case, as in the previous two theorems, we replace \( f_n(x) \) by \( S_N(t) \) and \( f(x) \) by \( S(t) \). Thus, we want the following:

- \( S_N(x) \rightarrow S(x) \) pointwise on \( I \)
- \( S'_N(x) \rightarrow G(x) \) uniformly on \( I \)
- \( S_N(x) \) is continuously differentiable for each \( N \)

and then we may conclude that \( S(x) \) is continuously differentiable with \( S'(x) = G(x) \). All we need is to formulate these requirements and result as follows:

**Theorem 6.3** Suppose that \( \sum_{k=0}^{\infty} u_k(x) \) satisfies the following conditions:

\[
\begin{align*}
\sum_{k=0}^{\infty} u_k(x) & \text{ converges pointwise on } I \\
\sum_{k=0}^{\infty} u'_k(x) & \text{ converges uniformly on } I \\
\text{\( u_k(x) \) is continuously differentiable for each } k
\end{align*}
\]

Then \( \sum_{k=0}^{\infty} u_k(x) \) is continuously differentiable and

\[
\frac{d}{dx} \left( \sum_{k=0}^{\infty} u_k(x) \right) = \sum_{k=0}^{\infty} u'_k(x).
\]

**Proof:** See *Kompendium om Konvergens*.  

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7 APPENDIX.

7.1 Supremum and Infimum: a recapitulation.

Definition 7.1 Let $A \subset \mathbb{R}$. Then the supremum of $A$, denoted by $\sup A$, is defined as the smallest number $a \in \mathbb{R}$ with the property that $x \leq a$ for all $x \in A$. In mathematical shorthand we have

$$\sup A = \min \{a \in \mathbb{R} : x \leq a \text{ for all } x \in A\}.$$

Similarly, the infimum of $A$, denoted by $\inf A$, is defined as the largest number $b \in \mathbb{R}$ with the property that $x \geq b$ for all $x \in A$. In mathematical shorthand we have

$$\inf A = \max \{b \in \mathbb{R} : x \geq b \text{ for all } x \in A\}.$$

Remark 7.1 Note that in these definitions neither the supremum nor the infimum need belong to the set $A$.

Example 7.1

(i) $A = [-1, 3]$. Here we have $\sup A = 3$, $\inf A = -1$, and both these belong to $A$.

(ii) $A = ]-1, 3]$. Here $\sup A = 3$, $\inf A = -1$, but only $\inf A$ belongs to $A$.

(iii) $A = ]-1, 3[$. Here $\sup A = 3$, $\inf A = -1$, and both are not in $A$.

(iv) $A = [-1, \infty]$. Here $\inf A = -1$ whereas $\sup A$ does not exist.

Definition 7.2 Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the supremum of $f(x)$ over $A$ is defined as the smallest number $a \in \mathbb{R}$ with the property that $f(x) \leq a$ for all $x \in A$. In mathematical shorthand we have

$$\sup_{x \in A} f(x) = \min \{a \in \mathbb{R} : f(x) \leq a \text{ for all } x \in A\}.$$

Similarly, the infimum of $f(x)$ over $A$ is defined as the largest number $b \in \mathbb{R}$ with the property that $f(x) \geq b$ for all $x \in A$. In mathematical shorthand we have

$$\inf_{x \in A} f(x) = \max \{b \in \mathbb{R} : f(x) \geq b \text{ for all } x \in A\}.$$

Example 7.2

(i) $f(x) = x^3$, $A = [-1, 3]$ Then, since $f(x)$ is a strictly increasing function, we have

$$\sup_{x \in A} f(x) = 27, \quad \inf_{x \in A} f(x) = -1.$$
(ii) \( f(x) = x^2, \ A = [-1, 3] \) Then note that \( f(x) = x^2 \) is not strictly increasing on this interval: it is decreasing on \([-1, 0]\) and then strictly increasing on \([0, 3]\). So we have

\[
\sup_{x \in [-1, 0]} f(x) = 1, \quad \inf_{x \in [-1, 0]} f(x) = 0
\]

and

\[
\sup_{x \in [0, 3]} f(x) = 9, \quad \inf_{x \in [0, 3]} f(x) = 0.
\]

Combining these two observations, we find that

\[
\sup_{x \in [-1, 3]} f(x) = 9, \quad \inf_{x \in [-1, 3]} f(x) = 0.
\]

(iii) \( f(x) = \arctan x, \ A = \mathbb{R} \). Here we have a strictly increasing function, and we have

\[
\sup_{x \in \mathbb{R}} f(x) = \frac{\pi}{2}, \quad \inf_{x \in \mathbb{R}} f(x) = -\frac{\pi}{2}.
\]

It is tempting to take the largest value of a function on an interval as the supremum, and the least value for the infimum. The last example shows that the neither the supremum nor the infimum need be attainable values of a function. However, we have the following simple but useful result:

**Lemma 7.1** Suppose that \( f(x) \) is a real-valued continuous function on the closed, bounded interval \([a, b]\). Then

\[
\sup_{x \in [a, b]} f(x) = \max\{f(x) : x \in [a, b]\}, \quad \inf_{x \in [a, b]} f(x) = \min\{f(x) : x \in [a, b]\}.
\]

That is, the supremum of a continuous function over a closed, bounded interval is equal to its largest value over that interval, and the infimum is the least value of the function over the interval.

**Proof:** Since \( f(x) \) is continuous and the interval is closed, then \( f(x) \) has a largest value and a least value on the interval: there exist \( x_1, x_2 \in [a, b] \) so that \( f(x_1) \leq f(x) \leq f(x_2) \) for all \( x \in [a, b] \), and we now see that

\[
\sup_{x \in [a, b]} f(x) = f(x_2), \quad \inf_{x \in [a, b]} f(x) = f(x_1),
\]

and the result is proved.