Transform theory 2023-06-03 – Solutions

1. (a) No. Note that $u(-t)$ is undefined if $t > 0$. Or if we extend $u(t) = 0$ for $t < 0$, then $\mathcal{L}(u(-t)) = 0 \neq U(-s)$ in general.
   (b) Yes. We have $\| \mathcal{F} u\|_\infty \leq \|u\|_{L^1(\mathbf{R})}$, so choose $C = \|u\|_{L^1(\mathbf{R})}$.
   (c) No. The function U is discontinuous at $\omega = 0$.
   (d) No. Note that $u(-\pi) = -\pi/2 \neq \pi/2 = u(\pi)$, so the periodic extension of u is discontinuous. The convergence of the Fourier series can not be uniform on $\mathbf{R}$.
   (e) No. The function u is not exponentially bounded. Indeed, for example using the root-test, we find that $R = \infty$ so the series is divergent for all z.

   Answer: No, Yes, No, No, No.

2. We take the Z transform of the equation and find that

$$z^2U(z) - z^2u[0] - zu[1] - U(z) = \frac{4z}{z+1},$$

where we assume that (at least) $|z| > 1$. Reformulating this equation, we find that

$$U(z)\left(z^2 - 1\right) = z^2 + z + \frac{4z}{z+1} \quad \Leftrightarrow \quad \begin{aligned} U(z) &= \frac{z^2+z}{z^2-1} + \frac{4z}{(z+1)(z^2-1)} \\ &= \frac{z}{z-1} + \frac{4z}{(z+1)^2(z-1)}. \end{aligned}$$

We decompose into partial fractions:

$$U(z) = \frac{z}{z-1} + z\left(\frac{-2}{(z+1)^2} + \frac{1}{z-1} - \frac{1}{z+1}\right) = \frac{2z}{z-1} - \frac{z}{z+1} - \frac{2z}{(z+1)^2}.$$

From a table:

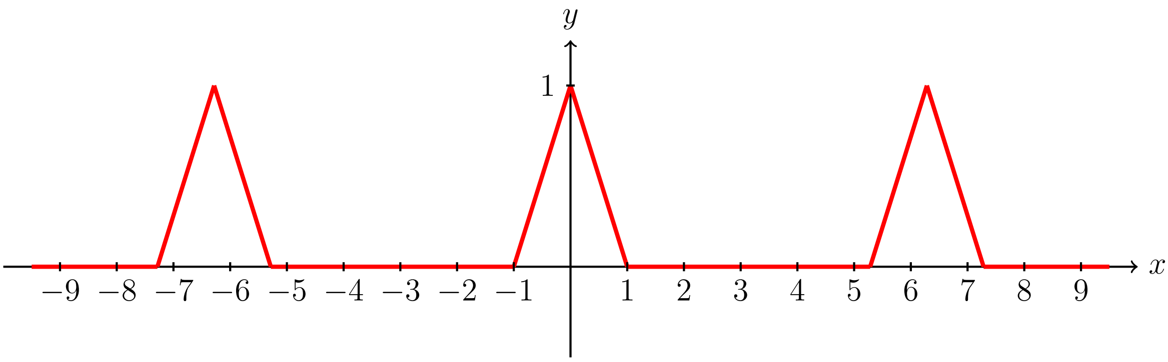
$$\mathcal{Z}(1) = \frac{z}{z-1}, \quad \mathcal{Z}\left((-1)^k\right) = \frac{z}{z+1}, \quad \mathcal{Z}\left(k(-1)^k\right) = \frac{-z}{(z+1)^2}.$$

By linearity and uniqueness, we therefore find that

$$u[k] = 2 - (-1)^k + 2k(-1)^k = 2 - (-1)^k + 2k(-1)^k, \quad k = 0, 1, 2, \ldots$$

Answer: $u[k] = 2 - (-1)^k + 2k(-1)^k$, $k = 0, 1, 2, 3, \ldots$

3. Clearly $u \in E$. This is clear since the function is piecewise “linear.” It is also clear that $D^{\pm}u(x)$ exists at all points. Moreover, u is continuous on $\mathbf{R}$ (u is continuous on $[-\pi, \pi]$, $u(-\pi) = u(\pi)$ and u is periodically extended). Hence – by Dirichlet’s theorem – the Fourier series of u is convergent and converges to $u(x)$ for all $x \in \mathbf{R}$. Moreover, $u' \in E$ (u' is piecewise constant), and therefore the convergence of the Fourier series is uniform on $\mathbf{R}$. We sketch the function below.



Since u is an even function, $b_k = 0$ for $k = 1, 2, \ldots$ For $k \geq 1$, we find (using the fact that u and $\cos kx$ are even functions)

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\cos kx\,dx = \frac{1}{\pi}\int_{-1}^{1}(1-|x|)\cos kx\,dx = \frac{2}{\pi}\int_{0}^{1}(1-x)\cos kx\,dx$$
$$= \frac{2}{\pi}\left(\left[(1-x)\frac{\sin kx}{k}\right]_0^1 + \int_0^1 \frac{\sin kx}{k}\,dx\right) = \frac{2}{\pi}\left[\frac{-\cos kx}{k^2}\right]_0^1 = \frac{2(1-\cos k)}{\pi k^2}$$

and

$$a_0 = \frac{1}{\pi}\int_{-1}^{1}(1-|x|)\,dx = \frac{1}{\pi}.$$

Hence

$$u(x) \sim \frac{1}{2\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1-\cos k}{k^2}\cos kx =: S(x).$$

By the argument above, the Fourier series $S(x)$ converges to $u(x)$, so $S(x) = u(x)$ for all $x \in \mathbf{R}$. In particular, we can let $x = 0$ to find that

$$1 = u(0) = \frac{1}{2\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1-\cos k}{k^2} \quad\Leftrightarrow\quad 1 - \frac{1}{2\pi} = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1-\cos k}{k^2}$$
$$\Leftrightarrow\quad \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1-\cos k}{k^2} = 1 - \frac{1}{2\pi}$$
$$\Leftrightarrow\quad \sum_{k=1}^{\infty}\frac{1-\cos k}{k^2} = \frac{\pi}{2}\left(1 - \frac{1}{2\pi}\right) = \frac{\pi}{2} - \frac{1}{4},$$

which was what we wanted to prove.

Answer: $u(x) \sim \frac{1}{2\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1-\cos k}{k^2}\cos kx$; see above.

4. The left-hand side is a convolution of f with the function $t \mapsto \frac{1}{t^2+4}$. The right-hand side is a translation of the function $t \mapsto \frac{1}{t^2+25}$. Taking the Fourier transform of the equation, we obtain that

$$F(\omega)\frac{\pi}{2}e^{-2|\omega|} = e^{i\omega}\frac{\pi}{5}e^{-5|\omega|} \quad\Leftrightarrow\quad F(\omega) = e^{i\omega}\frac{2}{5}e^{-3|\omega|} = e^{i\omega}\frac{6}{5\pi}\,\frac{\pi}{3}e^{-3|\omega|}.$$

Since $\mathcal{F}((t^2+9)^{-1}) = \frac{\pi}{3}e^{-3|\omega|}$, it follows from the translation property that

$$\mathcal{F}\left(\frac{1}{(t+1)^2+9}\right) = e^{i\omega}\,\frac{\pi}{3}\,e^{-3|\omega|}.$$

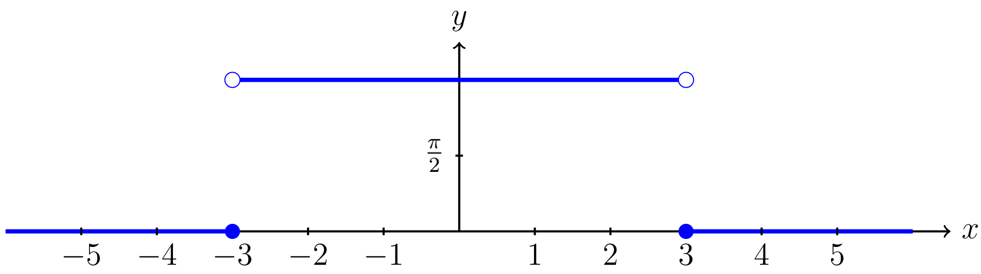
Therefore we have found

$$f(t) = \frac{6}{5\pi}\,\frac{1}{(t+1)^2+9}, \qquad t \in \mathbf{R}.$$

It is clear that $f \in G(\mathbf{R})$.

Answer: $f(t) = \frac{6}{5\pi}\,\frac{1}{(t+1)^2+9}$.

5. (a) We observe that $u \in G(\mathbf{R})$ and sketch the graph below.



Hence the Fourier transform exists and by definition (if $\omega \neq 0$)

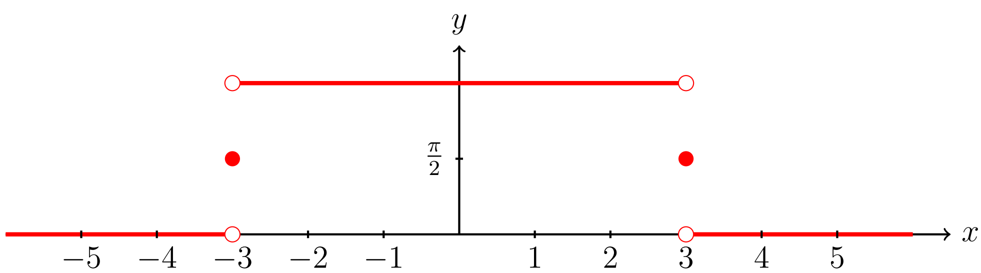
$$U(\omega) = \int_{-\infty}^{\infty} u(x)e^{-i\omega x}\,dx = \int_{-3}^{3} e^{-i\omega x}\,dx = \left[\frac{e^{-ix\omega}}{-i\omega}\right]_{x=-3}^{x=3}$$
$$= \frac{e^{i3\omega}-e^{-i3\omega}}{i\omega} = \frac{2\sin 3\omega}{\omega}.$$

At $\omega = 0$, we can either use the continuity of U to define $U(0)$ or calculate directly:

$$U(0) = \int_{-\infty}^{\infty} u(x)e^{-ix}\,dx = \int_{-3}^{3} dx = 6.$$

(b) First, we note that $\frac{\sin 3\omega}{\omega}$ is $\frac{1}{2}U(\omega)$ and that $D^{\pm}u(x)$ exists for all x, so by Dirichlet's theorem, we find that

$$\lim_{t\to\infty}\int_{-t}^{t}\frac{1}{2}U(\omega)e^{i\omega x}\,d\omega = \frac{2\pi}{2}\cdot\frac{u(x^+)+u(x^-)}{2} = \begin{cases} \pi, & |3| < \pi, \\ \pi\cdot(0+1)/2 = \frac{\pi}{2}, & x = \pm 3, \\ 0, & |x| > 3. \end{cases}$$



(c) We use the substitution $t = 3\omega$ and then Parseval's identity to find that

$$\int_{-\infty}^{\infty} \left(\frac{\sin\omega}{\omega}\right)^2 d\omega = \frac{1}{4}\int_{-\infty}^{\infty} \left(\frac{2\sin 3t}{3t}\right)^2 3\,dt = \frac{1}{12}\int_{-\infty}^{\infty} |U(t)|^2\,dt$$
$$= \frac{2\pi}{12}\int_{-\infty}^{\infty} |u(x)|^2\,dx = \frac{\pi}{6}\int_{-3}^{3} 1\,dx = \pi.$$

Answer: (a) $U(\omega) = \begin{cases} \dfrac{2\sin(3\omega)}{\omega}, & \omega \neq 0, \\ 6, & \omega = 0; \end{cases}$ (b) $\begin{cases} 2\pi, & |x| < 3, \\ \pi, & x = \pm 3, \\ 0, & |\xi| > 3; \end{cases}$ (c) π.

6. (a) We have a periodic function with $T = 2$ so

$$\mathcal{L}\,u(s) = \frac{1}{1-e^{-2s}}\int_0^2 u(t)e^{-st}\,dt = \frac{1}{1-e^{-2s}}\left(\int_0^1 -e^{-st}\,dt + \int_1^2 e^{-st}\,dt\right)$$
$$= \frac{1}{1-e^{-2s}}\left(\left[\frac{-e^{-st}}{-s}\right]_0^1 + \left[\frac{e^{-st}}{-s}\right]_1^2\right) = \frac{1}{1-e^{-2s}}\left(\frac{e^{-s}-1}{s} + \frac{-e^{-2s}+e^{-s}}{s}\right)$$
$$= \frac{e^{-2s}-2e^{-s}+1}{s(e^{-2s}-1)} = \frac{\left(e^{-s}-1\right)^2}{s(e^{-2s}-1)}, \quad \operatorname{Re} s > 0.$$

(b) We see that, if $U(s) = \mathcal{L}\,u(s)$ with $\operatorname{Re} s > b$, then

$$\mathcal{L}(u(at))(s) = \int_0^{\infty} u(at)e^{-st}\,dt = /\ y = at\ / = \int_0^{\infty} u(y)e^{-sy/a}\,\frac{dy}{a}$$
$$= \frac{1}{a}\int_0^{\infty} u(y)e^{-(s/a)y}\,dy = \frac{1}{a}\,\mathcal{L}\,u\left(\frac{s}{a}\right), \quad \operatorname{Re} s > ab. \qquad \square$$

Answer: (a) $\dfrac{\left(e^{-s}-1\right)^2}{s(e^{-2s}-1)}$ (b) see above.

7. Let $u_k(x) = \dfrac{1}{3x^4+5k^4}$. Clearly

$$|u_k(x)| \le \frac{1}{k^4}, \quad k = 1, 2, 3, \ldots$$

so the series defining $u(x)$ is convergent for all x (actually uniformly convergent by the M-test). To show that $u(x)$ is differentiably, we prove the uniform convergence of the series

$$\sum_{k=1}^{\infty} u_k'(x) = \sum_{k=1}^{\infty} \frac{-12x^3}{(3x^4+5k^4)^2}.$$

Clearly $u_k'(x) \to 0$ as $x \to \pm\infty$ and

$$u_k''(x) = \frac{180x^2(x^4-k^4)}{(3x^4+k^4)^3} \quad \Rightarrow \quad \left[\ u_k''(x) = 0 \quad \Leftrightarrow \quad x^4 = k^4 \text{ or } x = 0\ \right]$$

so $u_k''(x) = 0$ if $x = \pm k$ or $k = 0$. Clearly $u_k''(0) = 0$ and therefore the maximum of $|u_k'(x)|$ is found at $x = \pm k$. Thus

$$|u_k'(x)| \le |u_k'(\pm k)| = \frac{12k^3}{(8k^4)^2} = \frac{3}{16k^5}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^5} < \infty$, the M-test proves that $v(x) = \sum_{k=1}^{\infty} u_k'(x)$ is uniformly convergent. Moreover, u_k' are continuous for $k = 1, 2, 3, \ldots$, so v is a continuous function (by the uniform convergence). This is sufficient for claiming that u is differentiable with $u'(x) = v(x)$ for all x. Thus $u \in C^1$.

Answer: see above.