Transform theory 2023-08-17 – Solutions

1. (a) Yes. Since $z(U(z) - u[0]) = z \sum_{k=1}^{\infty} u[k]z^{-k} = \sum_{k=0}^{\infty} u[k+1]z^{-k}$, we see that if $|z| \to \infty$ then this expression tends to $u[1]$ by the initial value theorem.

   (b) Yes. The partial sums are obviously continuous functions so if the convergence is uniform, then the Fourier series converges to something that must be continuous.

   (c) No. For instance $u(x) = H(x+1) - H(x-1)$ has the transform $2\operatorname{sinc}\omega$ which does not belong to $L^1(\mathbf{R})$.

   (d) Nope. The function is discontinuous at $x = -3$ and $x = 2$, which is not possible for a Fourier transform of a function from G (which is uniformly continuous).

   (e) No, this is impossible. The exponent t^2 grows too fast (can't be bounded by kt for any $k > 0$).

   Answer: Yes. Yes. No. No. No.

2. We assume that u, u', u'' all belong to X_a (and verify this at the end). Taking the Laplace transform, we obtain that

$$s^2U(s) - su(0) - u'(0) - 2\big(sU(s) - u(0)\big) + 2U(s) = \frac{2(s-1)}{(s-1)^2+1}$$

$$\Leftrightarrow \quad U(s)\left(s^2 - 2s + 2\right) = \frac{2(s-1)}{(s-1)^2+1} + s, \quad \operatorname{Re} s > 1.$$

Hence,

$$U(s) = \frac{2(s-1)}{((s-1)^2+1)^2} + \frac{s}{(s-1)^2+1} = \frac{2(s-1)}{((s-1)^2+1)^2} + \frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}.$$

From a table, using the property $\mathcal{L}(e^{at}u(t)) = U(s-a)$,

$$\mathcal{L}(e^t \sin t) = \frac{2(s-1)}{(s-1)^2+1}, \quad \mathcal{L}(te^t \sin t) = \frac{2(s-1)}{((s-1)^2+1)^2}, \quad \mathcal{L}(e^t \cos t) = \frac{1}{(s-1)^2+1},$$

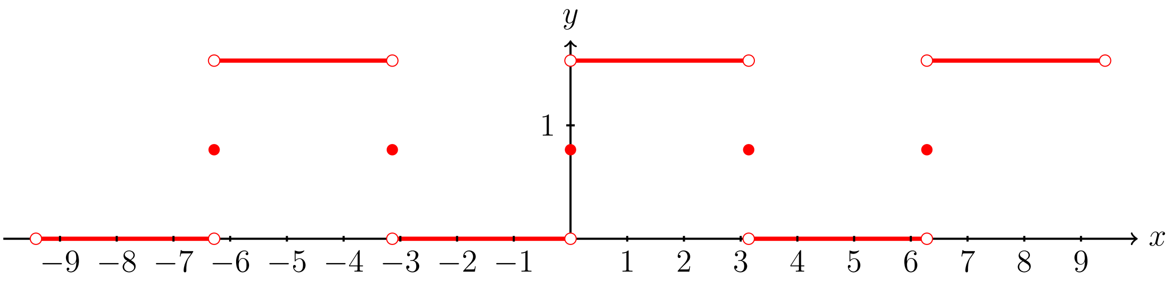
so

$$u(t) = te^t \sin t + e^t \sin t + e^t \cos t = e^t\big((t+1)\sin t + \cos t\big)$$

by uniqueness and linearity. Obviously u and its derivatives are exponentially bounded.

Answer: $u(t) = e^t\big((t+1)\sin t + \cos t\big), \quad t \geq 0.$

3. Clearly $u \in E$. This is clear since the function is piecewise "linear." It is also clear that $D^{\pm}u(x)$ exists at all points. Moreover, u is continuous if $x \neq n\pi$. Hence – by Dirichlet's theorem – the Fourier series of u is convergent and converges to $u(x)$ for all $x \neq n\pi$ and to $\pi/2$ if $x = n\pi$. Since the Fourier series converges to something that is discontinuous, the convergence cannot be uniform. We sketch the graph of the Fourier series below.



We find that, for $k \neq 0$,

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-ikx}\,dx = \frac{1}{2\pi}\int_{0}^{\pi} \pi e^{-ikx}\,dx = \frac{1}{2}\left[\frac{e^{-ikx}}{-ik}\right]_0^{\pi} \\ &= \frac{1}{i2k}\left(1-e^{ik\pi}\right) = \frac{1-(-1)^k}{i2k} \end{aligned}$$

and

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} \pi\,dx = \frac{\pi}{2}.$$

Hence

$$\begin{aligned} u(x) &\sim \frac{\pi}{2} + \sum_{k\neq 0} \frac{1-(-1)^k}{i2k}\,e^{ikx} = \frac{\pi}{2} + \sum_{k=-\infty}^{\infty} \frac{2}{i2(2k+1)}\,e^{i(2k+1)x} \\ &= \frac{\pi}{2} + \frac{1}{i}\sum_{k=-\infty}^{\infty} \frac{1}{2k+1}\,e^{i(2k+1)x}. \end{aligned}$$

Recall Parseval's identity, that is,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

We find that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx = \frac{1}{2\pi}\cdot\pi^3 = \frac{\pi^2}{2}$$

so

$$\frac{\pi^2}{2} = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{\pi^2}{4} + \sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2}.$$

Rearranging this yields

$$\frac{\pi^2}{2} - \frac{\pi^2}{4} = \sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2} \quad \Leftrightarrow \quad \sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{4}.$$

Answer: $u(x) = \frac{\pi}{2} + \frac{1}{i}\sum_{k=-\infty}^{\infty} \frac{1}{2k+1}\,e^{i(2k+1)x}$; see above.

4. Assuming that $y, y', y'' \in G$, we take the Fourier transform to find that

$$(i\omega)^2 Y(\omega) - 4Y(\omega) = \frac{6}{1+\omega^2} \quad \Leftrightarrow \quad -(4+\omega^2)Y(\omega) = \frac{6}{1+\imath\omega^2}$$
$$\Leftrightarrow \quad Y(\omega) = \frac{-6}{(1+\imath\omega^2)(1+4\omega^2)}$$

Decomposing into partial fractions, we find that

$$Y(\omega) = \frac{2}{\omega^2+4} - \frac{2}{\omega^2+1}.$$

From a table we find that

$$\mathcal{F}(e^{-2|x|}) = \frac{4}{4+\omega^2} \quad \text{and} \quad \mathcal{F}(e^{-|x|}) = \frac{1}{1+\omega^2},$$

so by linearity and uniqueness (on $G(\mathbf{R})$),

$$y(x) = \frac{1}{2}e^{-2|x|} - e^{-|x|}.$$

This function (and its derivatives) are absolutely integrable and continuous.

Answer: $y(x) = \dfrac{1}{2}e^{-2|x|} - e^{-|x|}$.

5. Consider the Z transform of $u[k] = k(k+1)a^k$, that is,

$$U(z) = \sum_{k=0}^{\infty} k(k+1)a^k z^{-k} = \sum_{k=0}^{\infty} k^2 a^k z^{-k} + \sum_{k=0}^{\infty} k a^k z^{-k}.$$

From a table we find that

$$U(z) = \frac{az}{(z-a)^2} + \frac{az^2+a^2z}{(z-a)^3} = \frac{az(z-a)+az^2+a^2z}{(z-a)^3} = \frac{2az^2}{(z-a)^3}, \quad |z| > |a|.$$

Since $|a| < 1$, we can consider $U(z=1)$:

$$U(z=1) = \sum_{k=0}^{\infty} k(k+1)a^k = \frac{2a}{(1-a)^3}.$$

Answer: $\dfrac{2a}{(1-a)^3}$.

6. (a) Let $u_n(t) = \dfrac{2+2n^2t}{n^2+n^2t^2+nt^4}$, $n = 1, 2, 3, \ldots$ and $0 \le x \le 1$. Then

$$u_n(t) = \frac{2/n^2+2t}{1+t^2+t^4/n} \to \frac{2t}{1+t^2},$$

as $n \to \infty$. Moreover,

$$\left|u_n(t) - \frac{2t}{1+t^2}\right| = \left|\frac{(2/n^2+2t)(1+t^2) - 2t(1+t^2+t^4/n)}{(1+t^2+t^4/n)(1+t^2)}\right|$$
$$= \left|\frac{2/n^2+2t^2/n^2-2t^5/n}{(1+t^2+t^4/n)(1+t^2)}\right| = \frac{1}{n}\left|\frac{2/n+2t^2/n-2t^5}{(1+t^2+t^4/n)(1+t^2)}\right|$$
$$\le \frac{1}{n}\frac{2+2+2}{(1+0+0)(1+0)} = \frac{6}{n}.$$

Clearly this means that

$$\sup_{0\le t\le 1}\left|u_n(t)-\frac{2t}{1+t^2}\right|\le \frac{6}{n}\to 0,$$

as $n\to\infty$. The convergence is therefore uniform and

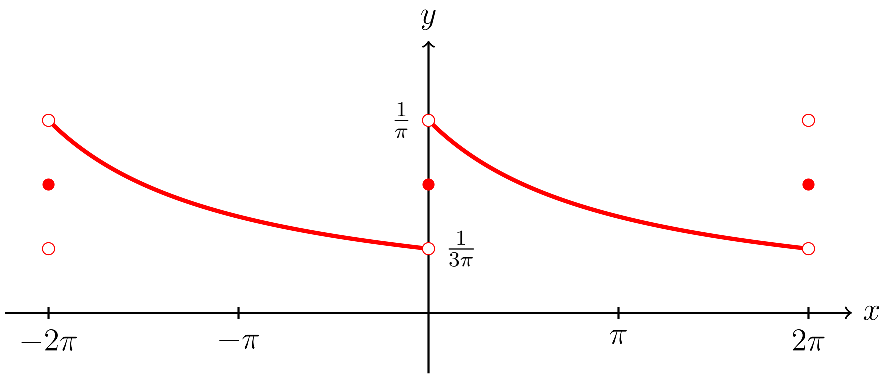
$$\lim_{n\to\infty}\int_0^1 \frac{2+2n^2t}{n^2+n^2t^2+nt^4}\,dt=\int_0^1 \lim_{n\to\infty}\frac{2+2n^2t}{n^2+n^2t^2+nt^4}\,dt=\int_0^1\frac{2t}{1+t^2}\,dt$$
$$=\left[\ln\left(1+t^2\right)\right]_0^1=\ln 2.$$

(b) We note that

$$\mathcal{L}(e^{at}u(t))(s)=\int_0^\infty u(t)e^{at}e^{-st}\,dt=\int_0^\infty u(t)e^{-(s-a)t}\,dt=(\mathcal{L}\,u)(s-a),$$

Answer: (a) $\dfrac{\left(e^{-s}-1\right)^2}{s(e^{-2s}-1)}$ (b) see above.

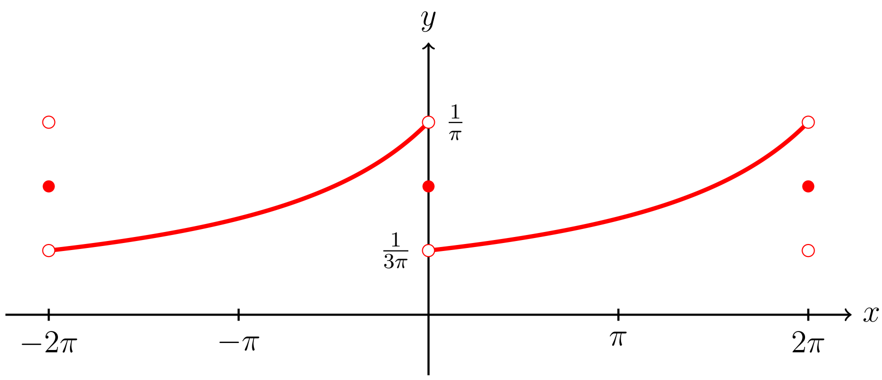
7. Let $f(t)$ be the periodic extension of $f(t)=\dfrac{1}{\pi+t}$, $0\le t<2\pi$. Since $f\in E'$, the Fourier series of f converges to $f(t)$ for $t\ne 2n\pi$ and to $\dfrac{1}{2}\left(\dfrac{1}{\pi}-\dfrac{1}{3\pi}\right)=\dfrac{2}{3\pi}$ when $x=2n\pi$.



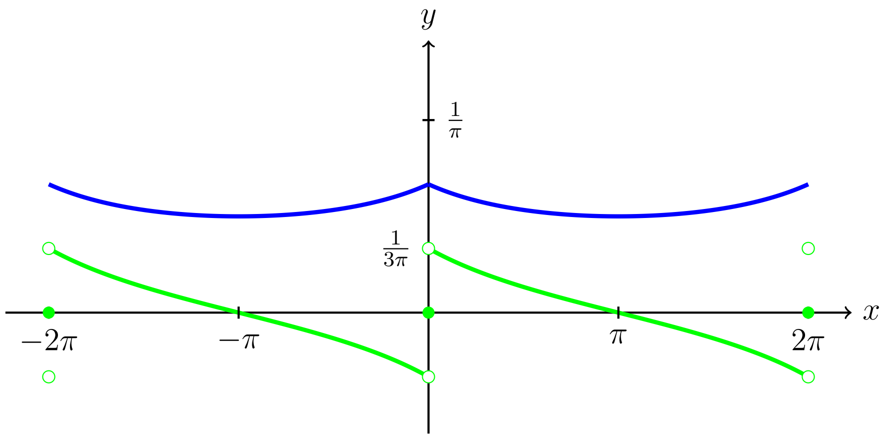
The integral in the question is the L^2-norm difference of f with a pure sine series, so lets split f into an odd and an even part: $f(t)=f_o(t)+f_e(t)$. One way is to define

$$f_o(t)=\frac{1}{2}\left(f(t)-f(-t)\right)\qquad\text{and}\qquad f_e(t)=\frac{1}{2}\left(f(t)+f(-t)\right).$$

Clearly $f_o(-t)=-f_o(t)$, $f_e(-t)=f_e(t)$ and $f_o(t)+f_e(t)=f(t)$ for $t\in\mathbf{R}$. To visualize what happens, consider first the Fourier series for $f(-t)$.



Clearly $f_o \in E'$ and $f_e \in E'$ (with f_e continuous as well). Therefore $f_o(t) = \sum_{k=1}^{\infty} d_k \sin kt$ for $t \neq 2n\pi$ (green) and $f_e(t) = \sum_{k=0}^{\infty} a_k \cos kt$ for $t \in \mathbf{R}$ (blue):



So what was the point of this? Considering that $\sin nt \perp \cos nt$ in L^2, we need to choose b_n as the Fourier coefficients of f_o. Then

$$\int_{-\pi}^{\pi} \left| f_e(t) + f_o(t) - \sum_{k=1}^{\infty} b_n \sin nt \right|^2 dt = \int_{-\pi}^{\pi} |f_e(t)|^2 \, dt = 2\int_0^{\pi} f_e(t)^2 \, dt$$

since f_e is even. Noting that for $0 < t < 2\pi$,

$$f_e(x) = \frac{1}{2}\left(\frac{1}{\pi + t} + \frac{1}{3\pi - t}\right),$$

we find that for $0 < t < 2\pi$,

$$\begin{aligned} |f_e(x)|^2 &= \frac{1}{4}\left(\frac{1}{\pi + t} + \frac{1}{3\pi - t}\right)^2 = \frac{1}{4}\left(\frac{1}{(\pi + t)^2} - \frac{2}{(t + \pi)(t - 3\pi)} + \frac{1}{(3\pi - t)^2}\right) \\ &= \frac{1}{4}\left(\frac{1}{(\pi + t)^2} + \frac{1}{2\pi}\left(\frac{1}{\pi + t} - \frac{1}{t - 3\pi}\right) + \frac{1}{(3\pi - t)^2}\right), \end{aligned}$$

so

$$\begin{aligned} 2\int_0^{\pi} f_e(x)^2 \, dx &= \frac{1}{2}\left[\frac{-1}{t + \pi} + \frac{1}{2\pi}\ln\left|\frac{t + \pi}{t - 3\pi}\right| + \frac{-1}{t - 3\pi}\right]_0^{\pi} = \frac{1}{2}\left(\frac{2}{3\pi} + \frac{1}{2\pi}\ln 3\right) \\ &= \frac{4 + 3\ln 3}{12\pi}. \end{aligned}$$

Answer: $\dfrac{4 + 3\ln 3}{12\pi}$.