Transform theory 2024-01-03 – Solutions

1. (a) Yes. Absolute convergence implies uniform convergence and the partial sums are continuous functions so the limit is a continuous function.
   (b) No, since (for example) $U(\omega) \to \dfrac{\pi}{2}$ as $\omega \to \infty$, the function doesn't satisfy the Riemann-Lebesgue lemma.
   (c) No, this is not true in general.
   (d) No. This is impossible since the Z transform has to have a limit as $|z| \to \infty$ (the initial value theorem; the limit is $u[0]$).
   (e) Yes. Any polynomial grows slower at infinity than any function e^{at} with $a > 0$.

   Answer: Yes. No. No. No. Yes.

2. Taking the Z transform with $|z| > 4$ yields

$$z^2U(z) - (z^2u[0] + zu[1]) - 5(zU(z) - zu[0]) + 6U(z) = \frac{z}{z-2} + \frac{z}{z-4}$$
$$\Leftrightarrow \quad \left(z^2 - 5z + 6\right) U(z) = 2z - z^2 + \frac{z(z-4) + z(z-2)}{(z-4)(z-2)}.$$

   Moreover, $z^2 - 5z + 6 = (z-3)(z-2)$, so

$$U(z) = \frac{2z - z^2}{(z-2)(z-3)} + \frac{2z^2 - 6z}{(z-4)(z-3)(z-2)^2} = \frac{-z}{z-3} + \frac{2z}{(z-4)(z-2)^2}.$$

   We decompose into partial fractions:

$$z \cdot \frac{2}{(z-4)(z-2)^2} = z\left(\frac{1/2}{z-4} - \frac{1/2}{z-2} - \frac{1}{(z-2)^2}\right).$$

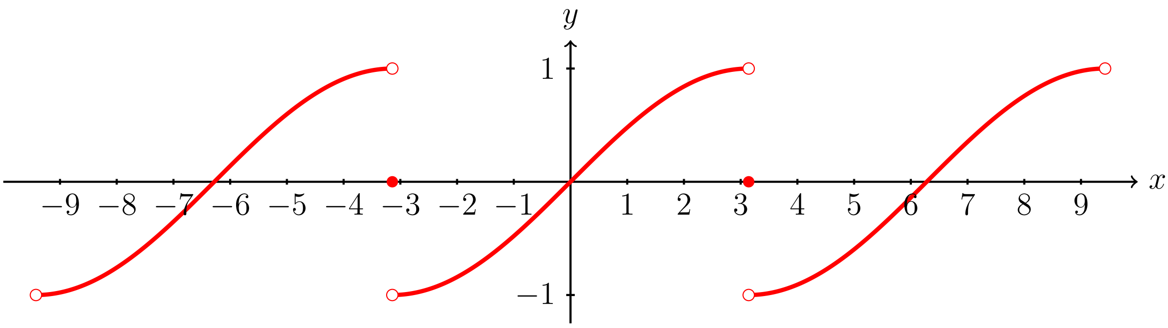
   By uniqueness, we use the table to find that

$$u[n] = -3^n + \frac{1}{2}\,4^n - \frac{1}{2}\,2^n - \frac{n}{2}2^n = 2^{2n-1} - 3^n - \frac{n+1}{2}\,2^n.$$

   We see that $u[0] = -1$ and $u[1] = -3$ and can directly verify that this solves the equation.

   Answer: $u[n] = 2^{2n-1} - 3^n - \dfrac{n+1}{2}\,2^n$, $n = 0, 1, 2, \ldots$

3. Clearly $u \in E$. This is clear since the function is smooth except for jump points at $x = (2n+1)\pi$, where right- and lefthand limits exist. It is also clear that $D^{\pm}u(x)$ exists at all points. Moreover, u is continuous if $x \neq (2n+1)\pi$. Hence – by Dirichlet's theorem – the Fourier series of u is convergent and converges to $u(x)$ for all $x \neq (2n+1)\pi$ and to 0 if $x = (2n+1)\pi$ (for example $u(\pi^-) = 1$ and $u(\pi^+) = -1$). Since the Fourier series converges to something that is discontinuous, the convergence cannot be uniform. We sketch the graph of the Fourier series below.



We need the Fourier coefficients, so we observe that u is an odd function so a pure sine-series is sufficient. The integrand is even so for $k \geq 1$,

$$
\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin\left(\frac{x}{2}\right)\sin kx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin\left(\frac{x}{2}\right)\sin kx\,dx \\
&= -\frac{1}{2\pi}\int_0^{\pi} \left(e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}\right)\left(e^{ikx} - e^{-ikx}\right)\,dx \\
&= -\frac{1}{2\pi}\int_0^{\pi} \left(e^{ix(k+\frac{1}{2})} - e^{ix(\frac{1}{2}-k)} - e^{-ix(\frac{1}{2}-k)} + e^{-ix(\frac{1}{2}+k)}\right)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} \left(\cos(\frac{1}{2}-k)x - \cos(k+\frac{1}{2})x\right)\,dx = \frac{1}{\pi}\int_0^{\pi} \left(\cos(k-\frac{1}{2})x - \cos(k+\frac{1}{2})x\right)\,dx \\
&= \frac{1}{\pi}\left[\frac{\sin(k-\frac{1}{2})x}{k-\frac{1}{2}} - \frac{\sin(k+\frac{1}{2})x}{k+\frac{1}{2}}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\sin(k-\frac{1}{2})\pi}{k-\frac{1}{2}} - \frac{\sin(k+\frac{1}{2})\pi}{k+\frac{1}{2}}\right) \\
&= \frac{1}{\pi}\left(\frac{-(-1)^k}{k-\frac{1}{2}} - \frac{(-1)^k}{k+\frac{1}{2}}\right) = \frac{(-1)^{k+1}}{\pi}\left(\frac{1}{k-\frac{1}{2}} + \frac{1}{k+\frac{1}{2}}\right) = \frac{(-1)^{k+1}}{\pi}\left(\frac{2k}{(k+\frac{1}{2})(k-\frac{1}{2})}\right) \\
&= \frac{8(-1)^{k+1}k}{\pi(4k^2-1)}.
\end{aligned}
$$

Thus $u(x) \sim \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}k}{\pi(4k^2-1)} \sin kx$.

Answer: $u(x) \sim \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}k}{\pi(4k^2-1)} \sin kx$; see above.

4. The left-hand side is a convolution of u with $t \mapsto \cos(2t)$, so taking the Laplace transform (assuming that $u \in X_a$) shows that

$$U(s)\frac{s}{s^2+4} = \frac{1}{2}\mathcal{L}(t^3+t^2) = \frac{1}{2}\left(\frac{6}{s^4} + \frac{2}{s^3}\right) = \frac{3}{s^4} + \frac{1}{s^3}, \quad \operatorname{Re} s > 0.$$

So if $\operatorname{Re} s > 0$, we find that

$$U(s) = \frac{(4+s^2)}{s}\left(\frac{3}{s^4} + \frac{1}{s^3}\right) = \frac{1}{s^2} + \frac{3}{s^3} + \frac{4}{s^4} + \frac{12}{s^5}.$$

Since $\mathcal{L}(t^m) = \frac{m!}{s^{m+1}}$, we find by uniqueness that

$$u(t) = t + \frac{3t^2}{2} + \frac{2t^3}{3} + \frac{t^4}{2},$$

which clearly is a function in X_a.

Answer: $u(t) = t + \frac{3t^2}{2} + \frac{2t^3}{3} + \frac{t^4}{2}$, $t > 0$.

5. We're looking for a solution to $y''(x) = -4y(x+\pi) + 4\sin 6x$, so obviously y must be (at least) differentiable. Hence y is continuous. This means that y'' must be continuous (since y solves the equation). Hence $y \in C^2$. Which means that $y'' \in C^2$, so $y \in C^4$ and so on. In other words, the solution must be very smooth.

   - $y \in C^3$ implies that the Fourier series of y, y' and y'' converges to $y(x)$, $y'(x)$ and $y''(x)$, respectively (by Dirichlet's theorem). So, let $y(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$.
   - y being 2π-periodical and $y' \in E$ means we can form the termwise derivative of y (with equality due to the first point):
   $$y'(x) = \sum_{k=-\infty}^{\infty} ikc_k e^{ikx}.$$
   - Similarly, $y'' \in E$, so (with equality since $y \in C^3$)
   $$y''(x) = \sum_{k=-\infty}^{\infty} -k^2 c_k e^{ikx}.$$

   Therefore, we can write

   $$\begin{aligned} y''(x) + 4y(x+\pi) = 4\sin 6x \quad &\Leftrightarrow \quad \sum_{k=-\infty}^{\infty} (-k^2 + 4e^{ik\pi}) c_k e^{ikx} = -2ie^{i6x} + 2ie^{-i6x} \\ &\Leftrightarrow \quad \sum_{k=-\infty}^{\infty} (-k^2 + 4(-1)^k) c_k e^{ikx} = -2ie^{i6x} + 2ie^{-i6x}. \end{aligned}$$

   For y to be a solution to the differential equation, we must therefore (by uniqueness) have:
   $$k^2 = 4(-1)^k \quad \text{or} \quad c_k = 0, \quad k \neq \pm 6.$$
   For odd k, $k^2 \neq 4(-1)^k$, and clearly $k = 0$ does not solve $k^2 = 4(-1)^k$. Thus $c_0 = 0$ and $c_{2k+1} = 0$. For $|k| > 2$, we obviously have $k^2 > 4$, so $k^2 \neq 4(-1)^k$ and $c_k = 0$ for $|k| > 2$ and $k \neq \pm 6$. For $k = \pm 2$ however, we have $k^2 = 4$, so $c_{\pm 2}$ are arbitrary. If $k = -6$, then
   $$(-36+4)c_{-6} = 2i \quad \Leftrightarrow \quad c_{-6} = \frac{2i}{-32} = -\frac{i}{16} = \frac{1}{16i}$$
   and if $k = 6$, then
   $$(-36+4)c_6 = -2i \quad \Leftrightarrow \quad c_6 = \frac{-2i}{-32} = \frac{i}{16} = -\frac{1}{16i}.$$
   Hence our solutions must have the form
   $$y(x) = c_{-2}e^{-i2x} + c_2 e^{i2x} + \frac{1}{16i}\left(e^{-i6x} - e^{i6x}\right) = A\cos 2x + B\sin 2x - \frac{1}{8}\sin 6x,$$
   where A and B are arbitrary constants.

   Answer: $y(x) = A\cos 2x + B\sin 2x - \frac{1}{8}\sin 6x$.

6. (a) We observe that $f \in G(\mathbf{R})$ so the Fourier transform exists and

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \int_{-1}^{1} xe^{-i\omega x}\,dx = \left[x\frac{e^{-i\omega x}}{-i\omega}\right]_{-1}^{1} + \frac{1}{i\omega}\int_{-1}^{1} e^{-i\omega x}\,dx \\ &= \left[x\frac{e^{-i\omega x}}{-i\omega} + \frac{e^{-i\omega x}}{\omega^2}\right]_{-1}^{1} = \frac{e^{-i\omega}}{-i\omega} + \frac{e^{-i\omega}}{\omega^2} - \left(\frac{e^{i\omega}}{i\omega} + \frac{e^{i\omega}}{\omega^2}\right) \\ &= i\frac{e^{-i\omega} + e^{i\omega}}{\omega} + i\frac{e^{-i\omega} - e^{i\omega}}{i\omega^2} = \frac{2i\cos\omega}{\omega} - \frac{2i\sin\omega}{\omega^2}, \quad \omega \neq 0. \end{aligned}$$

At $\omega = 0$, we calculate directly:

$$F(0) = \int_{-\infty}^{\infty} f(x)e^{-i\cdot 0\cdot x}\,dx = \int_{-1}^{1} x\,dx = 0.$$

(b) Note that with $F(\omega)$ from (a), for $\omega \neq 0$,

$$|F(\omega)|^2 = \left(\frac{2\cos\omega}{\omega} - \frac{2\sin\omega}{\omega^2}\right)^2 = 4\left(\frac{\cos\omega}{\omega} - \frac{\sin\omega}{\omega^2}\right)^2$$

and since $f \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, we can use Plancherel's theorem:

$$\int_{-\infty}^{\infty} \left(\frac{\cos\omega}{\omega} - \frac{\sin\omega}{\omega^2}\right)^2 d\omega = \frac{2\pi}{4}\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{\pi}{2}\int_{-1}^{1} x^2\,dx = \frac{\pi}{2}\cdot\frac{2}{3} = \frac{\pi}{3}.$$

Answer: (a) $F(\omega) = \dfrac{2i\cos\omega}{\omega} - \dfrac{2i\sin\omega}{\omega^2}$, $\omega \neq 0$, $F(0) = 0$ (b) see above.

7. Since

$$0 \leq \frac{1}{(x+k)^2} \leq \frac{1}{k^2}, \quad x \geq 0,\ k = 1, 2, 3, \ldots,$$

and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, it follows from Weierstrass M-test that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(x+k)^2}$ is uniformly convergent on $[0, 1]$. Thus we can exchange the order of integration and summation, yielding that

$$\begin{aligned} \int_0^1 \left(\sum_{k=1}^{\infty} \frac{1}{(x+k)^2}\right) dx &= \sum_{k=1}^{\infty} \int_0^1 \frac{1}{(x+k)^2}\,dx = \sum_{k=1}^{\infty} \left[-\frac{1}{x+k}\right]_0^1 = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 \end{aligned}$$

since the integrated series is a telescoping sum (write out a couple of terms to ensure this!).

Answer: 1.