Lecture 02: Linear Algebra

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Linear Spaces

A linear space V is a set such that addition and multiplication by scalars are defined and

$$u, v \in V \quad \Rightarrow \quad \alpha u + \beta v \in V, \quad \alpha, \beta \in \mathbb{C} \text{ (or } \mathbb{R}).$$

The operations addition and multiplication by constant behaves like we expect (associative, distributive and commutative). Multiplication depends on the elements.

Linear Combinations

Let $u_1, u_2, \ldots, u_n \in V$. We call

$$u = \sum_{k=1}^{n} \alpha_k u_k = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

a **linear combination**. If

$$\sum_{k=1}^{n} \alpha_k u_k = 0 \quad \Leftrightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0,$$

we say that $u_1, u_2, \ldots, u_n$ are **linearly independent**. The **linear span** $\text{span}\{u_1, u_2, \ldots, u_n\}$ of the vectors $u_1, u_2, \ldots, u_n$ is defined as the set of all linear combinations of these vectors (which is a linear space).

Examples

You've seen plenty of linear spaces before. One such example is the euclidian space $\mathbb{R}^n$ consisting of elements $(x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{R}$ and

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$
$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n).$$

Recall also that you've seen linear spaces that consisted of polynomials. The fact that our definition is general enough to cover many cases will prove to be very fruitful.

Function Spaces?

Let $C[a,b]$ be the linear space of all continuous functions $u\colon [a,b] \to \mathbb{C}$. This is indeed a linear space since

$$f, g \in C[a,b] \quad \Rightarrow \quad f+g \in C[a,b]$$

and

$$f \in C[a,b],\ \alpha \in \mathbb{C} \quad \Rightarrow \quad \alpha f \in C[a,b]$$

since the sum $((f+g)(x) = f(x) + g(x))$ of continuous functions is continuous and multiplying a continuous function by a constant $((\alpha f)(x) = \alpha f(x))$ is continuous.

Multiplication? Usually $(fg)(x) = f(x)g(x)$, so if $f, g \in C[a,b]$ then $fg \in C[a,b]$ since the product of continuous functions is continuous.

Absolutely Integrable Functions

The space $L^1(a,b)$ of absolutely integrable functions on $]a,b[$ is a linear space since

$$\int_a^b |f(x)|\,dx < \infty \text{ and } \int_a^b |g(x)|\,dx < \infty$$

$$\Rightarrow \quad \begin{aligned} \int_a^b |f(x)+g(x)|\,dx &\le \int_a^b \big(|f(x)|+|g(x)|\big)\,dx \\ &= \int_a^b |f(x)|\,dx + \int_a^b |g(x)|\,dx < \infty \end{aligned}$$

and

$$\int_a^b |\alpha f(x)|\,dx = |\alpha| \int_a^b |f(x)|\,dx < \infty.$$

Multiplication? Usually $(fg)(x) = f(x)g(x)$, but if $f,g \in L^1(a,b)$ then $fg \notin L^1(a,b)$ in general.

Basis

Basis

Definition. A subset $\{v_1, v_2, \ldots, v_n\} \subset V$ of linearly independent vectors is called a **base** for V if $V = \text{span}\{v_1, v_2, \ldots, v_n\}$ (meaning that every vector $v \in V$ can be expressed uniquely as a linear combination of the elements $v_1, v_2, \ldots, v_n$). The non-negative integer n is called the **dimension** of V: $\dim(V) = n$.

In general, however, we do not wish to restrict ourselves to finite dimensions or vectors of real or complex numbers.

So what if V is infinite dimensional? Meaning there is no finite set $\{v_1, v_2, \ldots, v_n\}$ that spans V?

We denote a sequence $u_1, u_2, u_3, \ldots$ (or $u_1, u_2, \ldots, u_n$ if it is a finite sequence) of elements of a linear space V by $(u_k)_{k=1}^{\infty}$ ($(u_k)_{k=1}^{n}$). If there's no risk of misunderstanding, we might just say "the sequence u_k."

As an example, consider the sequence $u_k = x + \dfrac{1}{k}$ in $\mathbb{R}$. That means that

$$u_1 = x + 1, \quad u_2 = x + \frac{1}{2}, \quad u_3 = x + \frac{1}{3}, \quad \ldots$$

We see that as $k \to \infty$, clearly $u_k \to x$. In other words, the sequence u_k *converges* to x. Reformulating a bit, what we have is that

$$\lim_{k\to\infty} |u_k - x| = \lim_{k\to\infty} \frac{1}{k} = 0.$$

Why write in this particular manner? We'll see...

Another example could be the decimal expansion of a real number:

$$x_1 = 3, \quad x_2 = 3.1, \quad x_3 = 3.14, \quad x_4 = 3.141, \quad x_5 = 3.1415, \quad \ldots,$$

where we might suspect that $x_n \to \pi$ as $n \to \infty$.
We've seen other examples of sequences in the form of partial sums like, for example,

$$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k, \quad n = 0, 1, 2, \ldots,$$

where

$$S_n = \frac{1 - (1/2)^{n+1}}{1 - 1/2} = 2\left(1 - 2^{-n-1}\right) \to 2 \text{ as } n \to \infty.$$

This was the reason for writing that $\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2.$

Divergent Sequences

Sequences do not need to be convergent. If $x_n = n$, $n = 0, 1, 2, \ldots$, for example, we obtain that

$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad \ldots,$$

so clearly $x_n \to \infty$. We might also consider something like $x_n = (-1)^n$ so that

$$x_0 = 1, \quad x_1 = -1, \quad x_2 = 1, \quad x_3 = -1, \quad \ldots,$$

where it is clear that x_n has no limit as $n \to \infty$ (but the sequence is bounded).

Sequences of Functions

Let

$$u_n(x) = n\left(\sin\left(x + \frac{1}{n}\right) - \sin(x)\right), \quad n = 1, 2, 3, \ldots$$

What happens as $n \to \infty$?
Another example would be $u_n(x) = \exp(nx)$, $n = 0, 1, 2, \ldots$.
As $n \to \infty$, we find that

$$u_n(x) \to \begin{cases} \infty, & x > 0 \\ 1, & x = 0, \\ 0, & x < 0. \end{cases}$$

Each u_n is continuous but the limiting function is not.

Series

We've also worked with function series previously (and obviously last lecture). The power series from TATA42 are an example:

$$S_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n, \quad n = 0, 1, 2, \ldots,$$

where we wrote

$$S(x) = \sum_{k=0}^{\infty} c_k x^k = \lim_{n \to \infty} S_n(x)$$

for those x where the series converges. That is, we *define* $S(x)$ as the limit of the sequence $S_n(x)$ as $n \to \infty$ whenever this limit exists.

Yet another example (last one, I promise..)

Consider the partial Fourier sums:

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx},$$

where c_k are the Fourier coefficients of some function $u\colon [-\pi, \pi] \to \mathbb{C}$. The main problem in the first half of this course is dealing with the sequence S_n of partial Fourier sums. In particular, we're interested in whether $S_n(x)$ converges (in some sense) and if the limit is actually $u(x)$ or not.

Continuity

Sequences are often useful when speaking of continuity of a function u from subsets of $\mathbb{R}$ or $\mathbb{C}$ into $\mathbb{R}$ or $\mathbb{C}$. I'm thinking of Heine's definition of continuity:

$$u \text{ is continuous at } x$$

if and only if

$$\lim_{n\to\infty} u(x_n) = u(x)$$

for all sequences x_n such that $\lim_{n\to\infty} x_n = x$, where we require that all x_n belongs to the domain of u.

Normed Linear Spaces

To measure distances between elements in a linear space (or "lengths" of elements), we define the abstract notion of a **norm** on a linear space (in the cases where this is allowed).

Norm

Definition. A normed linear space is a linear space V endowed with a norm $\| \cdot \|$ that assigns a non-negative number to each element in V in a way such that

1. $\|u\| \geq 0$ for every $u \in V$,
2. $\|\alpha u\| = |\alpha| \|u\|$ for $u \in V$ and every constant α,
3. $\|u + v\| \leq \|u\| + \|v\|$ for every $u, v \in V$.

Euclidian Distance

We note that in linear algebra, we typically used the norm

$$\|x\| := |\,(x_1, x_2, \ldots, x_n)\,| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

on the euclidean space $\mathbb{R}^n$ (or $\mathbb{C}^n$). We will use different types of norms in this course since we will be dealing with more complex spaces.

An element e in V with length 1, that is, $\|e\| = 1$, is called a **unit vector**.

Examples

Some examples of normed spaces

- The space $\mathbb{R}^n$ with the norm $\|(x_1, x_2, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.
- The space $\mathbb{R}^n$ with the norm $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$.

The first example is obviously already something you're familiar with. It is also an example of something we will call an inner product space below. The second example is a bit different. In some sense equivalent, but the norms yield different values for the same vector. Try to prove that the second one satisfies all the requirements for a norm.

The Sup-norm

The space of continuous functions with sup-norm

The space $C[a,b]$ consisting of continuous functions on the closed interval $[a,b]$ endowed with the norm

$$\|f\|_{C[a,b]} = \max_{a\leq t\leq b} |f(t)|, \quad f \in C[a,b].$$

Recall that a continuous function $|f|$ (if f is continuous then so is $|f|$) on a closed bounded interval $[a,b]$ always has a maximum value.

Sequence Spaces

Sequence spaces

The space l^1 consisting of all sequences $(x_1, x_2, x_3, \ldots)$ such that the norm

$$\|x\|_{l^1} = \sum_{k=1}^{\infty} |x_k| < \infty.$$

We might also consider the space l^p for $1 \leq p < \infty$ with the norm

$$\|x\|_{l^p} = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} < \infty.$$

The space L^1

The space of absolutely integrable functions

The space $L^1(\mathbb{R})$ of all integrable (on $\mathbb{R}$) functions with the norm

$$\|f\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

In other words, all functions that are absolutely integrable on $\mathbb{R}$. Note here that there's an army of dogs buried here. Indeed, the integral is not in the sense we're used to but rather in the form of the Lebesgue integral. We will not get stuck at this point, but it might be good to know.

Which Norm?

Exercise: Prove that the spaces above are normed linear spaces. Do you see any useful ways to consider some "multiplication" of vectors?

We see that an underlying linear space (like $\mathbb{R}^n$) might be endowed with different norms. This is true in general, and changing norms usually changes the results (at least for infinite dimensional spaces).

Convergence in Normed Spaces

Let $u_1, u_2, \ldots$ be a sequence in a normed space V. We say that $u_n \to u$ for some $u \in V$ if $\|u_n - u\| \to 0$ as $n \to \infty$. This is called **strong convergence** or **convergence in norm**. Note that we assumed above that the element u belonged to V. This may not be the case for every convergent sequence. How? Typically, we consider a linear subspace W of V. A sequence in W might be convergent when viewed as a sequence in V, but the limiting element might not belong to W.

A Peak at the Space E

Piecewise continuous function

Definition. A function $u\colon [a,b] \to \mathbb{C}$ is piecewise continuous if there are a finite number of points such that u is continuous everywhere except for at these points. Moreover, if $c \in]a,b[$ is one of these exception points, the limits

$$\lim_{x\to c^-} u(x) \quad \text{and} \quad \lim_{x\to c^+} u(x)$$

must exist.

The space $E[a,b]$ consists of all piecewise continuous functions on $[a,b]$.

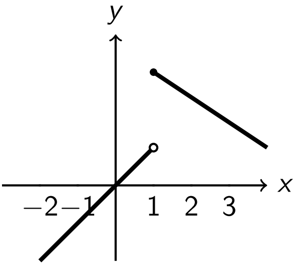
One-sided limits

We will denote the left- and righthand limits at a point c by

$$u(c^-) = \lim_{x \to c^-} u(x) \quad \text{and} \quad u(c^+) = \lim_{x \to c^+} u(x).$$

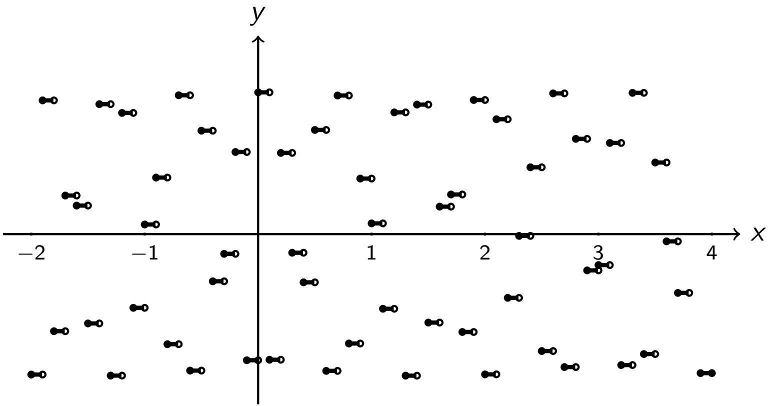
As an example, we could consider the function

$$f(x) = \begin{cases} x, & -2 \leq x < 1, \\ 4 - x, & 1 \leq x \leq 3. \end{cases}$$



A More Dramatic Example

The function below is in $E[-2, 4]$ (it is in fact even piecewise constant).



Continuous Mappings on Normed Spaces

Analogously with real analysis, we can define continuous mappings on normed spaces.

Continuity in normed spaces

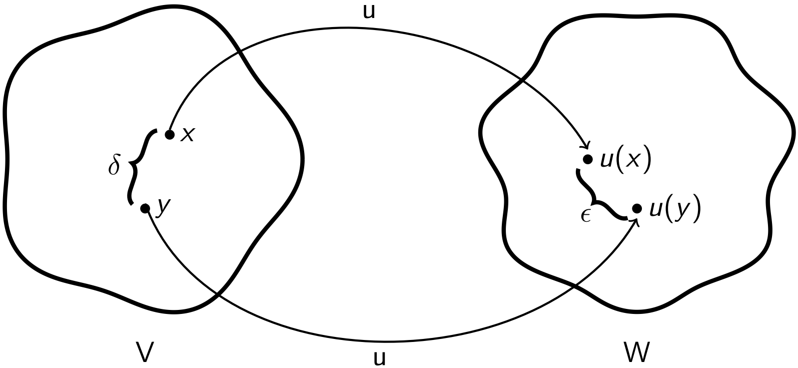
Definition. Let V and W be normed spaces. A function $u\colon V \to W$ is said to be continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$x, y \in V,\ \|x - y\|_V < \delta \quad \Rightarrow \quad \|u(x) - u(y)\|_W < \epsilon.$$

Basically this states that we call the function $u\colon V \to W$ continuous if x and y are close in V implies that $u(x)$ and $u(y)$ are close in W.

For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|x - y\|_V < \delta \quad \Rightarrow \quad \|u(x) - u(y)\| < \epsilon.$$



Series in Normed Spaces

Let $u_1, u_2, u_3, \ldots$ be a sequence in V. How do we interpret an expression of the form

$$S = \sum_{k=1}^{\infty} u_k, \tag{1}$$

that is, what does an infinite sum of elements in V mean? We define the partial sums by

$$S_n = \sum_{k=1}^{n} u_k, \quad n = 1, 2, 3, \ldots$$

If S_n converges to some $S \in V$ *in norm*, that is,

$$\lim_{n \to \infty} \left\| S - \sum_{k=1}^{n} u_k \right\| = 0,$$

then we write that (1) is convergent.

Absolute Convergence

Notice that this does *not* mean that

$$\sum_{k=1}^{\infty} \|u_k\| < \infty.$$

If this second series of real numbers is convergent, we call (1) **absolutely convergent** (compare with what we did in TATA42). Note also that an absolutely convergent series is convergent in the sense above (why?).

Inner Product Spaces

A norm is not enough to define a suitable geometry for our purposes, so we will usually work with inner product spaces instead.

Inner product

Definition. An inner product $\langle \cdot, \cdot \rangle$ on a vector space V is a complex valued (sometimes real) function on $V \times V$ such that

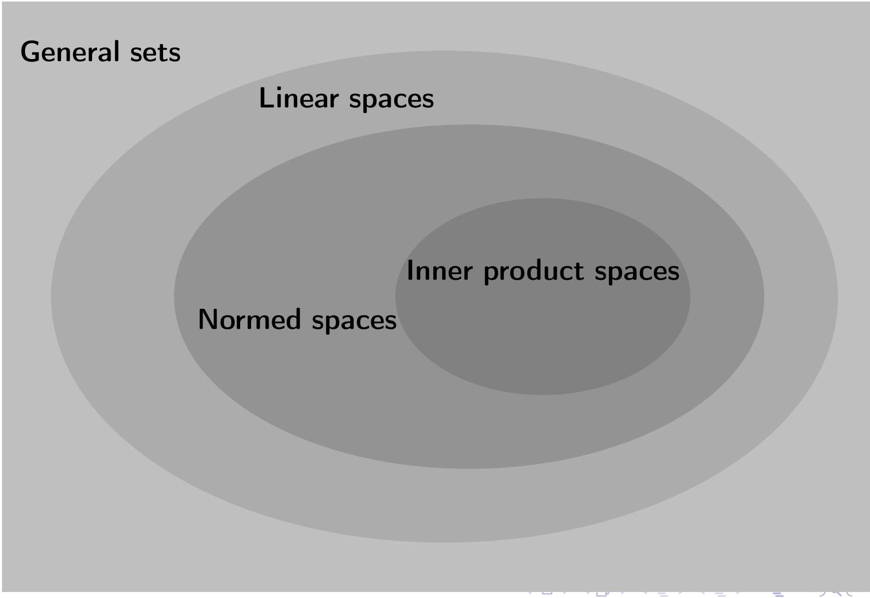
1. $\langle u, v \rangle = \overline{\langle v, u \rangle}$
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$
4. $\langle u, u \rangle \geq 0$
5. $\langle u, u \rangle = 0$ if and only if $u = 0$.

Properties

Note that 1 and 2 implies that $\langle u,\ v+w\rangle = \langle u,\ v\rangle + \langle u,\ w\rangle$ and that 1 and 3 implies that $\langle u,\ \alpha v\rangle = \overline{\alpha}\,\langle u,\ v\rangle$.

In an inner product space, we use $\|u\| = \sqrt{\langle u,\ u\rangle}$ as the norm. Why is this a norm? We'll get to that.

Notice that if we're given a linear space of functions, there's an infinite number of different inner products on this space that provides the same *geometry*. Suppose that $\langle u,\ v\rangle$ is an inner product. Then $\alpha\,\langle u,\ v\rangle$ is also an inner product for any $\alpha > 0$.


General sets
Linear spaces
Inner product spaces
Normed spaces

Examples

The inner product space $\mathbb{C}^n$

Definition. The space $\mathbb{C}^n$ consisting of n-tuples $(z_1, z_2, \dots, z_n)$ with

$$\langle z,\ w\rangle = \sum_{k=1}^{n} z_k \overline{w_k}, \quad z, w \in \mathbb{C}^n,$$

is an inner product space.

The Sequence Space l^2

The inner product space l^2

Definition. The space l^2 consisting of all sequences $(x_1, x_2, x_3, \ldots)$ of complex numbers such that the norm

$$\|x\|_{l^2} = \left(\sum_{k=1}^{\infty} |x_k|^2 \right)^{1/2} < \infty.$$

This is an inner product space if

$$\langle x,\ y \rangle = \sum_{k=1}^{\infty} x_k \overline{y_k}, \quad x, y \in l^2.$$

The Function Space of Square Integrable Functions

The inner product space $L^2(a, b)$

Definition. The space $L^2(a, b)$ consists of all "square integrable" functions with the inner product

$$\langle f,\ g\rangle = \int_a^b f(t)\overline{g(t)}\,dt.$$

Note that $a = -\infty$ and/or $b = \infty$ is allowed.

Normed Spaces vs. Inner Product Spaces

Why not the same examples as for the normed spaces? The simple answer is that most of those examples are *not* inner product spaces. The last two examples above are very important and the fact that it's the number 2 is not random and this is actually the only choice for when $L^p(a, b)$, which consists of functions for which

$$\|f\|_{L^p(a,b)} = \left(\int_a^b |f(t)|^p \, dt \right)^{1/p} < \infty$$

are inner product spaces. Again, we also note that the integrals above are more general than what we've seen earlier but if the function f is nice enough the value will coincide with the (generalized) Riemann integral.

Orthogonality

Orthogonality

Definition. If $u, v \in V$ and V is an inner product space, we say that u and v are orthogonal if $\langle u,\ v\rangle = 0$. We denote this by $u \perp v$.

A sequence u_n is called **pairwise orthogonal** if $\langle u_i,\ u_j\rangle = 0$ for every $i \neq j$. We have the generalized Pythagorean theorem.

Theorem. If $u_1, u_2, \ldots, u_n$ are pairwise orthogonal, then

$$\|u_1 + u_2 + \cdots + u_n\|^2 = \|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_n\|^2.$$

The Cauchy-Schwarz Inequality

The Cauchy-Schwarz inequality

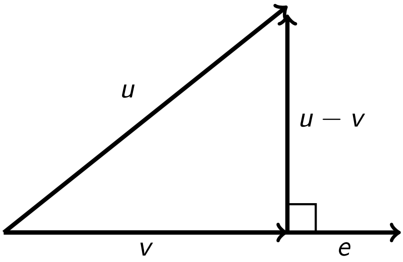
Theorem. If $u, v \in V$ and V is an inner product space, then

$$|\langle u,\ v\rangle| \leq \|u\|\|v\|.$$

Orthogonal Projection

Let $e \in V$ with $\|e\| = 1$. For $u \in V$, we define the **orthogonal projection** v of u on e by $v = \langle u,\ e \rangle e$. This is reasonable since $u - v \perp e$:

$$\langle u - v,\ e \rangle = \langle u,\ e \rangle - \langle v,\ e \rangle = \langle u,\ e \rangle - \langle u,\ e \rangle \langle e,\ e \rangle = 0.$$



Note that

$$\begin{aligned} \|u\|^2 &= \|u - v + v\|^2 \\ &= \|u - v\|^2 + \|v\|^2 \\ &= \|u - v\|^2 + |\langle u,\ e \rangle|^2. \end{aligned}$$

ON systems

ON system

Definition. Let V be an inner product space. We call

- $\{e_1, e_2, \ldots, e_n\} \subset V$,
- or $\{e_1, e_2, \ldots\} \subset V$,

an ON system in V if $e_i \perp e_j$ for $i \neq j$ and $\|e_i\| = 1$ for all i.

We do *not* assume that V is finite dimensional and that n is the dimension, and we do *not* assume that the ON system consists of finitely many elements.

Finite ON Systems

If the ON system is finite, consider $W = \text{span}\{e_1, e_2, \ldots, e_n\} \subset V$. We define the orthogonal projection Pv of a vector $v \in V$ onto the linear space W by

$$Pv = \sum_{k=1}^{n} \langle v,\ e_k \rangle\, e_k.$$

If $v \in W$, then clearly $Pv = v$. If $v \notin W$, then Pv is the vector that minimizes $\|v - Pv\|$. Note that this happens if $v - Pv \perp W$ (meaning perpendicular to every vector in W). We also note that

$$\|v\|^2 = \|v - Pv\|^2 + \sum_{k=1}^{n} |\langle v,\ e_k \rangle|^2$$

These facts are well-known from linear algebra.

Infinite ON Systems

If the ON system is infinite, let

$$P_n v = \sum_{k=1}^{n} \langle v,\ e_k \rangle\, e_k, \quad v \in V,\ n = 1, 2, 3, \ldots$$

Each $P_n v$ is the projection on a specific n-dimensional subspace of V (the order of the elements in the ON system is fixed).

Bessel's inequality

Theorem. Let V be an inner product space, let $v \in V$ and let $\{e_1, e_2, \ldots\}$ be an ON system in V. Then

$$\sum_{k=1}^{\infty} |\langle v,\ e_k \rangle|^2 \le \|v\|^2.$$

The Riemann-Lebesgue Lemma

Since $\|v\| < \infty$ for every $v \in V$, this inequality proves that the series in the left-hand side converges. A direct consequence of this is the Riemann-Lebesgue lemma.

The Riemann-Lebesgue Lemma

Theorem. Let V be an inner product space, let $v \in V$ and let $\{e_1, e_2, \ldots\}$ be an ON system in V. Then

$$\lim_{n\to\infty} \langle v,\ e_n \rangle = 0.$$

The Infinite Dimensional Case

If $\dim(V) = n$ and our ON system has n elements, then we know that we can always represent $v \in V$ as $v = \sum_{k=1}^{n} \langle v,\ e_k \rangle e_k$ (standard linear algebra). What happens if $\dim(V) = \infty$? When can we expect that an ON systems allows for something similar?

Closed ON Systems

Closed ON systems

Definition. Let V be an inner product space with $\dim(V) = \infty$. We call an orthonormal system $\{e_1, e_2, \ldots\} \subset V$ **closed** if for every $v \in V$ and every $\epsilon > 0$, there exists a sequence $c_1, c_2, \ldots, c_n$ of constants such that

$$\left\| v - \sum_{k=1}^{n} c_k e_k \right\| < \epsilon. \tag{2}$$

How do we typically find numbers c_k that work (they're not unique)? One answer comes in the form of orthogonal projections.

Generalized Fourier Coefficients

Fourier coefficients

Definition. For a given ON system, the complex numbers $\langle v,\ e_k\rangle$, $k = 1, 2, \ldots,$ are called the **generalized Fourier coefficients** of v.

We define the operator P_n that projects a vector onto the linear space spanned by $\{e_1, e_2, \ldots, e_n\}$ by

$$P_n v = \sum_{k=1}^{n} \langle v,\ e_k\rangle\, e_k, \quad v \in V.$$

Choosing c_k

We now note that the choice $c_k = \langle v,\ e_k \rangle$ is the choice that minimizes the left-hand side in (2). Indeed, suppose that $u = \sum_{k=1}^{n} c_k e_k$ for some constants c_k. Then

$$\begin{aligned}
\|v - u\|^2 &= \|v - P_n v + P_n v - u\|^2 \\
&= /\ (v - P_n v) \perp (P_n v - u)\ / = \|v - P_n v\|^2 + \|P_n v - u\|^2 \\
&= \|v - P_n v\|^2 + \left\| \sum_{k=1}^{n} (\langle v,\ e_k \rangle - c_k) e_k \right\|^2 \\
&= \|v - P_n v\|^2 + \sum_{k=1}^{n} |\langle v,\ e_k \rangle - c_k|^2.
\end{aligned}$$

In other words, $u = P_n v$ is the only element that minimizes $\|v - u\|$.

Alternative Definition of Closedness

Because of this, one can reformulate (equivalently) the definition of a closed ON system as follows.

Definition. Let V be an inner product space with $\dim(V) = \infty$. We call an orthonormal system $\{e_1, e_2, \ldots\} \subset V$ **closed** if for every $v \in V$

$$\lim_{n\to\infty} \left\| v - \sum_{k=1}^{n} \langle v,\ e_k\rangle\, e_k \right\| = 0.$$

We note that in the case where the ON system is closed, we can strengthen Bessel's inequality (by replacing the inequality with equality) obtaining what is known as Parseval's identity (or Parseval's formula).

Parseval's identity

Theorem. Suppose that $W = \{e_1, e_2, \ldots\}$ is an ON system for the inner product space V. Then W is closed if and only if **Parseval's identity** holds:

$$\sum_{k=1}^{\infty} |\langle v,\ e_k\rangle|^2 = \|v\|^2$$

for every $v \in V$.

Proof

Proof. Let $v \in V$. Then

$$\|v\|^2 = \|v - P_n v\|^2 + \|P_n v\|^2$$

since $v - P_n v \perp P_n v$. Hence

$$\left\| v - \sum_{k=1}^{n} \langle v,\ e_k \rangle e_k \right\|^2 = \|v\|^2 - \sum_{k=1}^{n} |\langle v,\ e_k \rangle|^2$$

and letting $n \to \infty$ in this equality, we see that closedness is equivalent with Parseval's identity holding. $\square$

Completeness

Definition. An ON-system $\{e_1, e_2, \ldots\}$ in V is called complete if, for every $v \in V$,

$$\langle v,\ e_k \rangle = 0 \text{ for all } k = 1, 2, 3, \ldots \quad \Leftrightarrow \quad v = 0.$$

We realize that completeness is something we want if we wish to use an ON-system as a *basis* for V since this is needed to make representations in terms of linear combinations of basis vectors needs to be unique to avoid problems.

Generalized Parseval's Identity

Generalized Parseval's identity

Theorem. Suppose that $\{e_1, e_2, e_3, \ldots\}$ is a closed infinite ON-system in V and let $u, v \in V$. If $a_k = \langle u,\ e_k\rangle$ and $b_k = \langle v,\ e_k\rangle$, then

$$\langle u,\ v\rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}.$$

Fourier Series?

So that brings us back to one of the main subjects of this course: Fourier series. Let's look at a particular inner product space.
We consider the space $L^2(-\pi, \pi)$ consisting of square integrable functions $u\colon [-\pi, \pi] \to \mathbb{C}$:

$$\int_{-\pi}^{\pi} |u(x)|^2\,dx < \infty.$$

We define the inner product on this space by

$$\langle u,\ v\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\overline{v(x)}\,dx.$$

Note that this infers that we have the norm

$$\|u\| = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\,dx\right)^{1/2},$$

which is finite for $u \in L^2(-\pi, \pi)$.

ON Systems in E (or $L^2(-\pi, \pi)$)

The set of functions e^{ikx}, $k \in \mathbb{Z}$, is a closed orthonormal system in E with the inner product defined above. We consider E as a subspace of $L^2(-\pi, \pi)$. Clearly we have

$$\|e^{ikx}\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ikx}\, dx = \frac{2\pi}{2\pi} = 1.$$

Similarly, if $k, l \in \mathbb{Z}$ and $k \neq l$, we have

$$\left\langle e^{ikx},\ e^{ilx} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ilx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-l)x}\, dx = 0$$

since $e^{i(k-l)x}$ is 2π-periodic. So this is an ON-system in E.

Closedness

The fact that this ON-system is closed is a more difficult argument so we'll get back to this in lecture 5. Note though, that E is a a subspace of $L^2(-\pi, \pi)$ that has some issues. For example there are sequences in E that converge (in the L^2-norm) to elements outside of E. This is a disadvantage, but nothing that will cause too many problems for us.

The Real System

The set of functions $\dfrac{1}{\sqrt{2}}$, $\cos kx$, $k = 1, 1, 2, \ldots$, $\sin kx$, $k = 1, 2, 3, \ldots$, is a closed orthonormal system in E with the inner product

$$\langle u,\ v\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x)\overline{v(x)}\,dx.$$

Note that the *normalization constant* is different compared to the complex case (why do you think that is?). We should observe that these two systems are equivalent due to Euler's formulas.

The Space E

So the question right now is *what do we really need?*
Most of the results we're going to see have a more general and complete version, but we would need considerably more time to develop the necessary tools to attack these problems. So what we're going to do instead is to consider the space E with the inner product

$$\langle u,\ v\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\overline{v(x)}\,dx, \quad u, v \in E. \tag{3}$$

As stated above, this space has some serious drawbacks, but these problems are not crucial to what we're going to do.
First, let's verify that things work as expected. When we write E, we now mean the combination of the set E of piecewise continuous functions combined with the inner product defined by (3).

Linear Space

E is a linear space. Obviously, if $u \in E$ and α is a constant, then αu has the same exception points as u (unless $\alpha = 0$) and the right- and lefthand limits will exist for $\alpha u(x)$. Let $u, v \in E$ and let $a_1, a_2, \ldots, a_n$ be the exceptions points of u and $b_1, b_2, \ldots, b_m$ be the exception points of v. Then $u + v$ has (at most) $m + n$ exception points. Indeed, if we sort the exception points as $c_1 < c_2 < \cdots < c_{n+m}$, then $u + v$ will be continuous on each $]c_i, c_{i+1}[$ and the right- and lefthand limits at the exception points will exist since either it is an exception point for u or v (potentially both), or it is a point of continuity for u or v. Therefore the limit of the sum exist.

Inner Product

Equation (3) *defines an inner product on* E. Most of the properties follow from the linearity of the integral. The fact that $\langle u,\ u\rangle = 0$ implies that $u = 0$ is clear since

$$\langle u,\ u\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(x)|^2\, dx = 0$$

so $u = 0$ is the only possible piecewise continuous function (if $u(x_0) \neq 0$ at some point then there is an interval $]x_0 - \delta, x_0 + \delta[$ where $|u(x)| > 0$ and so the Riemann integral will be strictly greater than zero).

Fourier Coefficients and the Riemann Lebesgue Lemma

So in general, we know that $\langle u,\ e_k \rangle \to 0$ as $k \to \infty$ if $\{e_1, e_2, \dots\}$ is an ON system with respect to the inner product at hand (in our case (3)). This was a consequence of Bessel's inequality. In particular, this means that for $u \in E$, we have

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} u(x) e^{-inx}\, dx = 0.$$

Note that this implies that

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} u(x) \sin(nx)\, dx = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{-\pi}^{\pi} u(x) \cos(nx)\, dx = 0.$$

So apparently these limits hold for all piecewise continuous functions. However, these identities are also true for $u \in L^1(-\pi, \pi)$ (this needs a proof however).