Exercises for the course Graph Theory TATA64

Mostly from Textbooks by Bondy-Murty (1976) and Diestel (2006)

Notation

$E(G)$ set of edges in G.

$V(G)$ set of vertices in G.

K_n complete graph on n vertices.

$K_{m,n}$ complete bipartite graph on $m + n$ vertices.

G^{c} the complement of G.

$L(G)$ line graph of G.

$c(G)$ number of components of G (Note: $\omega(G)$ in Bondy-Murty).

$o(G)$ number of odd components in G (i.e. number of components with an odd number of vertices.)

$d_G(v)$ degree of a vertex v in G.

$N_G(v)$ set of neighbors in G of a vertex v.

$\delta(G)$ minimum degree in G.

$\Delta(G)$ maximum degree in G.

$\alpha(G)$ independence number of G, i.e., the size of the largest independent set in G.

$\beta(G)$ minimum size of a vertex cover in G.

$\alpha'(G)$ size of a maximum matching in G.

$\beta'(G)$ minimum size of an edge cover in G.

$d_G(u, v)$ distance between u and v, i.e., length of a shortest path between u and v

$\kappa(G)$ connectivity of G, i.e. the greatest k such that G is k-connected.

$\kappa'(G)$ edge-connectivity of G, i.e. the greatest k such that G is k-edge-connected. (Note: $\lambda(G)$ in Diestel)

$\chi(G)$ chromatic number of G, i.e. minimum k such that G has a proper k-coloring.

$\chi'(G)$ chromatic index (edge-chromatic number) of G, i.e. minimum k such that G has proper k-edge coloring.

$\omega(G)$ clique number of G, i.e. the size of a maximum clique in G.

1 Basics. Trees.

1.1. Show that if G is a graph with $|V(G)| = n$, then $|E(G)| \leq \binom{n}{2}$, with equality if and only if G is complete.

1.2. Show that $|E(K_{m,n})| = mn$. Moreover, show that if G is bipartite, then $|E(G)| \leq \frac{|V(G)|^2}{4}$.

1.3. The k-cube Q_k is the graph whose vertices are the ordered k-tuples of 0's and 1's, two vertices being joined by an edge if and only if they differ in exactly one coordinate. Show that $|V(Q_k)| = 2^k$, $|E(G)| = k2^{k-1}$, and that Q_k is bipartite.

1.4. (a) The *complement* G^{c} of a graph G is the graph with vertex set $V(G)$, two vertices being adjacent in G^{c} if and only if they are not adjacent in G. Describe the graphs K_n^{c} and $K_{m,n}^{\mathsf{c}}$.

(b) G is *self-complementary* if $G \cong G^{\mathsf{c}}$. Show that if G is self-complementary, then $|V(G)| \equiv 0, 1$ mod 4.

1.5. Show that

(a) every induced subgraph of a complete graph is complete;

(b) every subgraph of a bipartite graph is bipartite.

1.6. Show that if a k-regular bipartite graph with $k > 0$ has a bipartition (X, Y), then $|X| = |Y|$.

1.7. Show that, in any group of two or more people, there are always two with exactly the same number of friends inside the group.

1.8. If a multigraph G has vertices $v_1, v_2, \ldots, v_n$, the sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is called a *degree sequence* of G. Show that a sequence $(d_1, d_2, \ldots, d_n)$ of non-negative integers, such that $d_1 \geq d_2 \geq \cdots \geq d_n$, is a degree sequence of some multigraph (loops not allowed) if and only if $\sum_{i=1}^n d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$.

1.9. A sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a (simple) graph with degree sequence $\mathbf{d}$. Show that the sequences $(7, 6, 5, 4, 3, 3, 2)$ and $(6, 6, 5, 4, 3, 3, 1)$ are not graphic.

1.10. Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of non-negative integers.

(a) Show that $\mathbf{d}$ is graphic if and only if $(d_2 - 1, d_3 - 1, \ldots d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphic.

(Hint: To prove necessity, first show that if $u_1v_1, u_2v_2 \in E(G)$ and $u_1v_2, u_2v_1 \notin E(G)$, then $G - \{u_1v_1, u_2v_2\} + \{u_1v_2, u_2v_1\}$ has the same degree sequence as G. Using this, show that if $\mathbf{d}$ is graphic, then there is a graph H such that $V(H) = \{v_1, v_2, \ldots v_n\}$, $d(v_i) = d_i$ for each $i = 1, \ldots, n$, and v_1 is adjacent to $v_2, \ldots, v_{d_1+1}$. The graph $H - v_1$ has degree sequence $(d_2 - 1, d_3 - 1, \ldots d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$.)

(b) Using (a), describe an algorithm for constructing a graph with degree sequence $\mathbf{d}$, if such a graph exists.

1.11. Show that a graph G contains a spanning bipartite subgraph H such that $d_H(v) \geq \frac{1}{2}d_G(v)$ for all $v \in V(G)$. (Hint: Show that a bipartite subgraph with the largest possible number of edges has this property.)

1.12. Show that if there is a (u, v)-walk (i.e. a walk beginning at u and ending at v) in G, then there is also a (u, v)-path in G.

1.13. (a) Show that if G is a n-vertex graph with $\delta(G) > \lfloor n/2 \rfloor - 1$, then G is connected.

(b) Find a disconnected $(\lfloor n/2 \rfloor - 1)$-regular graph for even n.

1.14. Show that if G is disconnected, then G^{c} is connected.

1.15. (a) Show that if $e \in E(G)$, then $c(G) \leq c(G-e) \leq c(G)+1$.

(b) Let $v \in V(G)$. Show that $G-e$ cannot, in general, be replaced by $G-v$ in the above inequality.

1.16. Show that if G is a connected graph and every degree in G is even, then, for any $v \in V(G)$, $c(G-v) \leq \frac{1}{2}d_G(v)$.

1.17. Show that any two longest paths in a connected graph have a vertex in common.

1.18. If vertices u and v are connected by a path in G, the *distance* between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G; if there is no path connecting u and v we define $d_G(u,v)$ to be infinite. Show that, for any three vertices u, v and w, $d(u,v) + d(v,w) \geq d(u,w)$.

1.19. The *diameter* of G is the maximum distance between two vertices of G. Show that if G has diameter greater than three, then G^{c} has diameter less than three.

1.20. Show that if G is a graph with diameter two, and $\Delta(G) = |V(G)| - 2$, then $|E(G)| \geq 2|V(G)| - 4$.

1.21. Show that if G is a connected non-complete graph, then G has three vertices u, v, w such that $uv, vw \in E(G)$ and $uw \notin E(G)$.

1.22. Show that if an edge e is in a closed trail of G, then e is in a cycle of G.

1.23. Show that if G is a graph with $\delta(G) \geq 2$, then G contains a cycle of length at least $\delta(G)+1$.

1.24. Show that the minor relation $\preccurlyeq$ defines a partial ordering on any set of graphs.

1.25. Prove that if a graph G contains a subdivision of a graph H as a subgraph, then H is a minor of G.

1.26. Is there an eulerian graph G with $|V(G)|$ even and $|E(G)|$ odd? Proof or counterexample!

1.27. Show that if G has no vertices of odd degree, then there are edge-disjoint cycles $C_1, C_2, \ldots, C_m$ such that $E(G) = E(C_1) \cup E(C_2) \cup \cdots \cup E(C_m)$.

1.28. Show that if a connected graph G has $2k > 0$ vertices of odd degree, then there are k edge-disjoint trails $Q_1, Q_2, \ldots, Q_k$ in G such that $E(G) = E(Q_1) \cup E(Q_2) \cup \cdots \cup E(Q_k)$.

1.29. Prove or disprove that every connected graph contains a walk that traverses every edge exactly twice.

1.30. Let G be a (simple) graph.

(a) Prove that the number of edges in $L(G)$ is $\sum_{v \in V(G)} \binom{d_G(v)}{2}$.

(b) Prove that G is isomorphic to $L(G)$ if and only if G is 2-regular.

1.31. Let M be the incidence matrix and A the adjacency matrix of a graph G.

(a) Show that every column sum of M is 2.

(b) What are the column sums of A?

1.32. (a) Show that if any two vertices of a graph G are connected by a unique path, then G is a tree.

(b) Prove that the endpoints of a longest path in a nontrivial (i.e. containing at least two vertices) tree both have degree one.

1.33. (a) Show that if G is a tree with $\Delta(G) \geq k$, then G has at least k vertices of degree one.

(b) Deduce that every tree with exactly two vertices of degree one is a path.

1.34. Let G be graph with $|V(G)|-1$ edges. Show that the following tree statements are equivalent:

(a) G is connected;

(b) G is acyclic;

(c) G is a tree.

1.35. Show that a sequence $(d_1, d_2, \ldots, d_n)$ of positive integers is a degree sequence of a tree if and only if $\sum_{i=1}^{n} d_i = 2(n-1)$. (Hint: Use e.g. induction on n)

1.36. Let T be an arbitrary tree on $k+1$ vertices. Show that if G is a graph with $\delta(G) \geq k$, then G has a subgraph isomorphic to T.

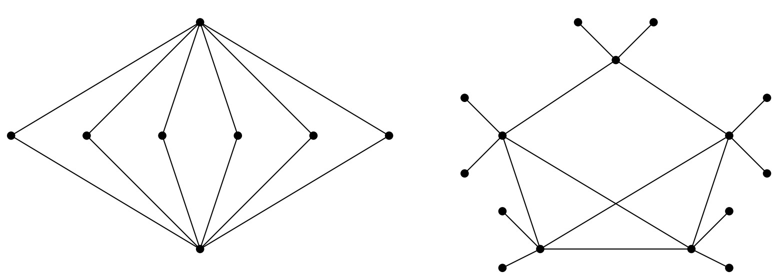
1.37. Show that if G is a multigraph and has exactly one spanning tree T, then $G = T$.

1.38. Lef F be a maximal forest of G (i.e. a subgraph of G such that $F + e$ is not a forest for any $e \in E(G) \setminus E(F)$). Show that

(a) for every component H of G, $F \cap H$ is a spanning tree of H;

(b) $|E(F)| = |V(G)| - c(G)$.

1.39. Find the number of nonisomorphic spanning tress in the following graphs.



1.40. Show that

(a) if every degree in G is even, then G has no cut edge;

(b) if G is a k-regular bipartite graph with $k \geq 2$, then G has no cut edge.

1.41. Let G be a connected graph with at least 3 vertices. Show that

(a) if G has a cut edge, then G has a vertex v such that $c(G-v) > c(G)$;

(b) the converse of (a) is not necessarily true.

1.42. Show that a graph that has exactly two vertices which are not cut vertices is a path.

1.43. Show that if e is an edge of K_n, then the number of spanning trees of $K_n - e$ is $(n-2)n^{n-3}$.

2 Matchings, factors, independent sets and covers

2.1. (a) Show that every k-cube has a perfect matching ($k \geq 2$).

(b) Find the number of different perfect matchings in K_{2n} and $K_{n,n}$.

2.2. Show that a tree has at most one perfect matching.

2.3. Let M be a matching in a bipartite graph G. Show that if M is not maximum, then G contains an augmenting path with respect to M.

2.4. (a) Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

(b) Let G be a bipartite graph. Prove that $\alpha(G) = |V(G)|/2$ if and only G has a perfect matching.

2.5. For each $k > 1$, find an example of a k-regular multigraph without loops that has no perfect matching. Also, find a cubic (simple) graph without a perfect matching.

2.6. Two people play a game on a graph G by alternately selecting distinct vertices $v_0, v_1, v_2, \ldots$ such that, for $i > 0$, v_i is adjacent to v_{i-1}. The last player able to select a vertex wins. Show that the first player has a winning strategy if and only if G has no perfect matching.

2.7. (a) For $S \subseteq V(G)$, define $N(S) = \cup_{v \in S} N(v)$. Show that a bipartite graph G has a perfect matching if and only if $|N(S)| \geq |S|$ for all $S \subseteq V(G)$.

(b) Give an example to show that the above statement does not remain valid if the condition that G be bipartite is dropped.

2.8. For $k > 0$, show that

(a) every k-regular bipartite graph is 1-factorable.

(b) every $2k$-regular graph is 2-factorable, i.e., it is the edge-disjoint union of 2-factors.

2.9. Let $A_1, A_2, \ldots, A_m$ be subsets of a set S. A *system of distinct representatives* for the family $(A_1, A_2, \ldots, A_m)$ is a subset $\{a_1, a_2, \ldots, a_m\}$ of S such that $a_i \in A_i$, $1 \leq i \leq m$ and $a_i \neq a_j$ for $i \neq j$. Show that $(A_1, A_2, \ldots A_m)$ has a system of distinct representatives if and only if $|\bigcup_{i \in J} A_i| \geq |J|$ for all subsets J of $\{1, 2, \ldots, m\}$.

2.10. Let G be a k-regular graph, with $|V(G)|$ even, that remains connected when any $k-2$ edges are deleted. Prove that G has a 1-factor.

2.11. A graph G is *factor-critical* if each subgraph $G - v$ obtained by deleting one vertex has a 1-factor. Prove that G is factor-critical if and only if $|V(G)|$ is odd and $o(G - s) \leq |S|$ for all nonempty $S \subseteq V(G)$.

2.12. A *permutation matrix* P is a $0, 1$-matrix having exactly one 1 in each row and column. Prove that a square matrix of nonnegative integers can be expressed as the sum of k permutation matrices if and only if all row sums and column sums equal k.

2.13. (a) Show that G is bipartite if and only if $\alpha(H) \geq \frac{1}{2}|V(H)|$ for every subgraph H of G.

(b) Show that G is bipartite if and only if $\alpha(H) = \beta'(H)$ for every subgraph H of G such that $\delta(H) > 0$.

2.14. A graph is α-*critical* if $\alpha(G - e) > \alpha(G)$ for all $e \in E(G)$. Show that a connected α-critical graph has no cut-vertices.

2.15. For every graph G, prove that $\beta(G) \leq 2\alpha'(G)$. For each $k \in \mathbf{N}$, construct a graph with $\alpha'(G) = k$ and $\beta(G) = 2k$.

2.16. Let G be a bipartite graph with parts X and Y, such that $|N(S)| \geq |S|$ whenever $\emptyset \neq S \subseteq X$. Prove that every edge of G belongs to some matching that saturates X.

3 Connectivity. Menger's theorem

3.1. (a) Show that if G is k-edge connected, with $k > 0$, and if E' is a set of k edges of G, then $c(G - E') \leq 2$.

(b) For all integers $k > 0$, find a k-connected graph G and a set V' of k vertices of G such that $c(G - V') > 2$.

3.2. Show that if a graph G is k-edge-connected, then $|E(G)| \geq k|V(G)|/2$.

3.3. (a) Show that if G is a graph and $\delta(G) \geq |V(G)| - 2$, then $\kappa(G) = \delta(G)$.

(b) Find a simple graph G with $\delta(G) = |V(G)| - 3$ and $\kappa(G) < \delta(G)$.

3.4. Show that if G is a graph and $\delta(G) \geq \lfloor |V(G)|/2 \rfloor$, then $\kappa'(G) = \delta(G)$, and prove that this is best possible by constructing for each $n \geq 4$ an n-vertex graph with $\delta(G) = \lfloor n/2 \rfloor - 1$ and $\kappa'(G) < \delta(G)$.

3.5. Show that if G is a cubic graph, then $\kappa'(G) = \kappa(G)$.

3.6. Give an example to show that if P is a path from u to v in a 2-connected graph G, then G does not necessarily contain a path Q from u to v that is internally disjoint from P.

3.7. Show that the block graph of any connected graph is a tree.

3.8. Show that if G has no even cycles, then each block of G is either K_1 or K_2 or an odd cycle.

3.9. Let G be a k-connected graph, and let S, T be disjoint subsets of $V(G)$ with size at least k. Prove that G has k pairwise disjoint S,T-paths (i.e. a collection of paths the origins of which all lie in S, and whose termini all lie in T).

3.10. Let G be a connected graph in which for every edge e, there are cycles C_1 and C_2 containing e whose only common edge is e. Prove that G is 3-edge-connected. Use this to show that the Petersen graph is 3-edge-connected.

3.11. Prove that a connected graph is k-edge-connected if and only if each of it blocks is k-edge-connected

3.12. Let $k \geq 2$. Show that a k-connected graph with at least $2k$ vertices has a cycle of length at least $2k$.

4 Vertex colorings. Planar graphs. Turan's theorem

4.1. Show that if G is a graph where any two odd cycles have a vertex in common, then $\chi(G) \leq 5$.

4.2. Prove that every graph G has a vertex ordering relative to which the greedy coloring algorithm uses $\chi(G)$ colors.

4.3. Prove that every k-chromatic graph has at least $\binom{k}{2}$ edges.

4.4. For every $n > 1$, find a bipartite graph on $2n$ vertices, ordered in such a way that the greedy coloring algorithm uses n rather than 2 colors.

4.5. Show that the only 2-critical graph is K_2, and the only 3-critical graphs are the odd cycles.

4.6. Prove that every triangle-free (i.e. not containing a cycle with 3 vertices) n-vertex graph has chromatic number at most $2\sqrt{n}$. (So every k-chromatic triangle-free graph has at least $k^2/4$ vertices.) (Hint: Iteratively remove a largest independent set from a triangle-free graph until it has "sufficiently" small maximum degree.)

4.7. A graph G is *vertex-color-critical* if $\chi(G - v) < \chi(G)$ for all $v \in V(G)$.

(a) Prove that every color-critical graph is vertex-color-critical.

(b) Prove that every 3-chromatic vertex-color-critical graph is color-critical.

4.8. Let G be a claw-free graph (i.e. no induced subgraph of G is isomorphic to $K_{1,3}$).

(a) Prove that the subgraph induced by the union of any two color classes in a proper coloring of G consists of paths and even cycles.

(b) Prove that if G has a proper coloring using exactly k colors, then G has a proper k-coloring where the color classes differ in size by at most one.

4.9. Let $G_3, G_4, \ldots,$ be the graphs obtained from $G_2 = K_2$ using Mycielski's construction. Show that each G_k is k-critical.

4.10. Show that $K_{3,3}$ is nonplanar.

4.11. (a) Show that $K_5 - e$ is planar for any edge e of K_5.

(b) Show that $K_{3,3} - e$ is planar for any edge e of $K_{3,3}$.

4.12. Show that a graph is planar if and only if each of its blocks is planar.

4.13. A plane graph is *self-dual* if it is isomorphic to its dual.

(a) Show that if G is self-dual, then $|E(G)| = 2|V(G)| - 2$.

(b) For each $n \geq 4$, find a self-dual plane graph on n vertices.

4.14. Let G be a plane graph. Show that $(G^*)^*$ is isomorphic to G (i.e. the dual of the dual of G is isomorphic to G) if and only G is connected.

4.15. A *plane triangulation* is a plane graph in which each face has degree three. Show that every plane graph is a spanning subgraph of some planar triangulation (if the graph has at least 3 vertices).

4.16. The *girth* of a graph is the length of its shortest cycle.

(a) Show that if G is a connected planar (simple) graph with girth $k \geq 3$, then $|E(G)| \leq k\frac{|V(G)|-2}{k-2}$.

(b) Using (a), show that the Petersen graph is nonplanar.

4.17. (a) Show that if G is a planar graph with at least 11 vertices, then G^{c} is nonplanar.

(b) Find a planar graph G with 8 vertices, such that G^{c} is also planar.

4.18. Show that if G is a plane triangulation, then $|E(G)| = 3|V(G)| - 6$.

4.19. Show, using Kuratowski's theorem, that the Petersen graph is non-planar.

4.20. What does the planar dual of a plane tree look like?

4.21. Wagner proved in 1937 that that the following condition is necessary and sufficient for a graph G to be planar: neither K_5 nor $K_{3,3}$ can be obtained from G by performing deletions and contractions of edges.

(a) Show that deletion and contraction of edges preserve planarity, and conclude that Wagner's conditions is necessary.

(b) Use Kuratowski's theorem to prove that Wagner's theorem is sufficient.

4.22. Use the four color theorem to prove that every planar graph is the edge-disjoint union of two bipartite graphs.

4.23. Derive the four color theorem from Hadwiger's conjecture for the case of graphs with chromatic number at least 5.

4.24. Prove that a graph is a complete multipartite graph if and only if it has no 3-vertex induced subgraph with one edge.

4.25. (a) Show that if G is a graph and $|E(G)| > |V(G)|^2/4$, then G contains a triangle.

(b) Find a graph G with $|E(G)| = \lfloor |V(G)|^2/4 \rfloor$ that contains no triangle.

(c) Show that if G is a non-bipartite graph and $|E(G)| > (|V(G)| - 1)^2/4 + 1$, then G contains a triangle.

Hint for (c): Assume that G contains no triangle, and consider a shortest odd cycle C in G. Show that each vertex in $V(G) \setminus V(C)$ can be joined to at most two vertices of C, and apply (a) to $G - V(C)$ to obtain a contradiction.

4.26. The Turan graph $T_{n,r}$ is the complete r-partite with b partite sets of size $a+1$ and $r-b$ partite sets of size a, where $a = \lfloor n/r \rfloor$ and $b = n - ra$.

(a) Prove that $|E(T_{n,r})| = (1 - 1/r)n^2/2 - b(r - b)/(2r)$.

(b) Show that if G is a complete r-partite graph on n vertices, then $|E(G)| \leq |E(T_{n,r})|$ with equality if and only if G is isomorphic to $T_{n,r}$.

4.27. Prove that every n-vertex graph with no $(r+1)$-clique has at most $(1-1/r)n^2/2$ edges. (Hint: Use the fact that a sum of squares $f = a_1^2 + a_2^2 + \cdots + a_k^2$, such that $a_1 + a_2 + \cdots + a_k = a$, is minimized when $a_i = a/k$ for all i.)

4.28. Let G be an n-vertex graph with m edges.

(a) Prove that $\omega(G) \geq \lceil n^2/(n^2 - 2m) \rceil$. (Hint: Use the previous exercise.)

(b) Prove that $\alpha(G) \geq \lceil n/(d+1) \rceil$, where d is the average degree of G. (Hint: use part (a).)

5 Edge Colorings. Hamilton cycles.

5.1. Show, by finding an appropriate edge coloring, that $\chi'(K_{m,n}) = \Delta(K_{m,n})$.

5.2. Show that the Petersen graph has chromatic index 4.

5.3. (a) Show that if G is bipartite, then G is contained in a $\Delta(G)$-regular bipartite graph.

(b) Using (a) and the fact that every regular bipartite graph has a 1-factor, give an alternative proof of König's edge coloring theorem.

5.4. Show that if G is bipartite with $\delta(G) > 0$, then G has a $\delta(G)$-edge coloring (not necessarily proper!) such that all $\delta(G)$ colors are represented at each vertex.

5.5. Show by finding appropriate edge colorings, that $\chi'(K_{2n-1}) = \chi'(K_{2n}) = 2n - 1$.

5.6. (a) Show that if G is a non-empty regular graph with $|V(G)|$ odd, then $\chi'(G) = \Delta(G) + 1$.

(b) Show that if G is a regular graph with a cut vertex, then $\chi'(G) > \Delta(G)$.

5.7. (a) Show that if G is a (loopless) multigraph, then G is contained in a Δ-regular (loopless) multigraph.

(b) Using (a) and Petersen's result that every $2k$-regular multigraph has a 2-factor, prove that $\chi'(G) \leq 3\Delta(G)/2$ for any (loopless) multigraph G with even maximum degree.

5.8. Apply Brooks' theorem (not Vizing's) to an 'appropriate' graph to prove that if G is a graph with $\Delta(G) = 3$, then $\chi'(G) \leq 4$.

5.9. Show that if either

(a) G is not 2-connected, or

(b) G is bipartite with bipartition (X, Y) where $|X| \neq |Y|$,

then G is not hamiltonian.

5.10. Prove that if G has a Hamilton path, then $o(G - S) \leq |S| + 1$, for every proper subset S of V.

5.11. A graph G is called uniquely k-edge-colorable if any two proper k-edge colorings of G induce the same partition of E. Show that every uniquely 3-edge-colorable 3-regular graph is hamiltonian.

5.12. Let G be a n-vertex graph that is not a forest and contains no cycles of length less than 5. Prove that the complement of G is hamiltonian. (Hint: First use Ore's condition on G^c to deduce that there are vertices x, y such that $xy \in E(G)$ and $d_G(x) + d_G(y) \geq n - 1$.)

5.13. Let G be a connected graph with $\delta(G) = k \geq 2$ and $|V(G)| > 2k$.

(a) Let P be a maximal path in G (i.e. not a subgraph of any longer path). Prove that if $|V(P)| \leq 2k$, then the induced subgraph $G[V(P)]$ has a spanning cycle.

(b) Use part (a) to prove that G has a path with at least $2k + 1$ vertices.

5.14. A graph is hypohamiltonian if G is not hamiltonian but $G - v$ is hamiltonian for every $v \in V(G)$. Show that the Petersen graph is hypohamiltonian.

6 Ramsey theory

6.1. Determine the Ramsey number $R(3,3)$.

6.2. Let R_n denote the Ramsey number $R(K_3^{(1)}, K_3^{(2)}, \ldots, K_3^{(n)})$, where each $K_3^{(i)}$ is a triangle (i.e. this Ramsey number is the value of r such that n-edge-coloring K_r forces a monochromatic triangle).

(a) Show that $R_n \leq n(R_{n-1} - 1) + 2$.

(b) Noting that $R_2 = 6$, use (a) to show that $R_n \leq \lfloor n!e \rfloor + 1$.

(c) Deduce that $R_3 \leq 17$.

6.3. Determine the Ramsey number $R(K_{1,m}, K_{1,n})$. (Hint: The answer depends on whether m and n are even or odd.)

6.4. Let $G_1, G_2, \ldots, G_m$ be graphs. The *generalized Ramsey number* $R(G_1, G_2, \ldots, G_m)$ is the smallest integer n such that every m-edge coloring of K_n contains, for some i, a subgraph isomorphic to G_i in color i. Show that $R(P_4, P_4) = 5$, $R(P_4, C_4) = 5$, and $R(C_4, C_4) = 6$, where P_4 is a 4-vertex path, and C_4 is a 4-vertex cycle.

6.5. Prove that if T is a tree on m vertices, and $m-1$ divides $n-1$, then $R(T, K_{1,n}) = m + n - 1$.

6.6. Prove that $R(mK_2, mK_2) = 3m - 1$, where mK_2 is the graph consisting of m pairwise disjoint copies of K_2.