TATA66 Exercises, first set

Hand in solutions to six of the following ten exercises

1 Let $f_n\colon \mathbb{R} \to [0,\infty]$, $n \geq 1$, be measurable. Show, using the monotone convergence theorem, that

$$\int_{\mathbb{R}} \Big(\sum_{n=1}^{\infty} f_n(x) \Big)\, dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)\, dx.$$

(Do not worry about measurability of sets and functions in this or other exercises.)

2 Use the dominated convergence theorem to show that $\displaystyle\lim_{n\to\infty} \int_0^{\infty} \frac{e^{-nx}}{\sqrt{x} + \sin x}\, dx = 0.$

***3** Let $f_k \in L^1(\mathbb{R})$, $k \geq 1$, be such that $\|f_k\|_1 \leq 2^{-k}$.

(a) Let $g(x) = \sum_{k=1}^{\infty} |f_k(x)|$, $x \in \mathbb{R}$. Show that $\int_{\mathbb{R}} g(x)\, dx < \infty$.

Hence $g(x) < \infty$ for almost every $x \in \mathbb{R}$. Let $A = \{x \in \mathbb{R}\colon g(x) = \infty\}$ and let $f\colon \mathbb{R} \to \mathbb{C}$ be the function given by $f(x) = 0$, $x \in A$, and $f(x) = \sum_{k=1}^{\infty} f_k(x)$, $x \notin A$ (the series being absolutely convergent for every $x \notin A$).

(b) Show that $f \in L^1(\mathbb{R})$ and that $\sum_{k=1}^{n} f_k \to f$ in $L^1(\mathbb{R})$ as $n \to \infty$.

(The result in this exercise is an essential step in proving that $L^1(\mathbb{R})$ is complete.)

4 Let $f_k \to f$ in $L^1(\mathbb{T})$. Show that $\widehat{f_k}(n) \to \widehat{f}(n)$ for every $n \in \mathbb{Z}$.

5 Prove Corollary 3.4.11 in *Fourier Analysis, Distribution Theory, and Wavelets.*

***6** For $N \geq 0$ let $P_N(t) = \int_0^t D_N(r)\, dr$, where D_N is the Dirichlet kernel.

(a) Show that P_N is odd, that $0 \leq P_N(t) \leq 2\pi$ if $0 \leq t \leq \pi$, and that the function $P_N(t) - t$ is 2π-periodic.

(b) Let $f \in C^1(\mathbb{T})$. Show that $|S_N f(t)| \leq |\widehat{f}(0)| + 2\pi \|f'\|_1$ for all $t \in \mathbb{R}$.

7 Let X be the linear space $C([0,1])$ with inner product given by $(f,g) = \int_0^1 f(x)\overline{g(x)}\, dx$. Show that X is not a Hilbert space by giving a concrete example of a Cauchy sequence $f_1, f_2, \ldots$ in X such that there is no $f \in X$ such that $f_n \to f$ in X.

***8** The linear space $X = C^1([0,1])$ with norm given by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ is a Banach space. Let $Y = \{f \in X\colon f(0) = f'(0) = 0\}$ and let $g(x) = x$, $0 \leq x \leq 1$. Show that Y is a closed subspace of X, that $\inf_{f\in Y} \|g - f\| = 1$, but that $\|g - f\| > 1$ for every $f \in Y$.

9 Show that $\{\sin \frac{t}{4}, \sin \frac{3t}{4}, \sin \frac{5t}{4}, \sin \frac{7t}{4}, \ldots\}$ is an orthogonal basis for $L^2([0, 2\pi])$.

***10** Let $f, g \in L^2(\mathbb{T})$.

(a) Show that the n:th Fourier coefficient of $(S_N f)(S_N g)$ is equal to zero if $|n| > 2N$ and equal to $\sum_{k=a}^{b} \widehat{f}(n-k)\widehat{g}(k)$ if $|n| \leq 2N$ (determine a and b).

(b) Show that $fg \in L^1(\mathbb{T})$ and that $(S_N f)(S_N g) \to fg$ in $L^1(\mathbb{T})$.

(c) For $n \in \mathbb{Z}$, show that the series $\sum_{k=-\infty}^{\infty} \widehat{f}(n-k)\widehat{g}(k)$ is absolutely convergent and, using the results in (a), (b), and exercise 4, that its sum is equal to $\widehat{fg}(n)$.