TATA66 Exercises, second set

Hand in solutions to six of the following ten exercises

1 For $n = 1, 2, \ldots$, let $f_n = 2^{n-1}\chi_{[-1/2^n,\, 1/2^n]}$ and let $p_n = f_1 * f_2 * \ldots * f_n$.
(a) Roughly sketch the graphs of p_1, p_2, and p_3 (no calculation required).
(b) Show that $\operatorname{supp} p_n \subseteq [-1, 1]$ for all $n \geq 1$.
(c) Show that $p_n \in C^{n-2}(\mathbb{R})$ if $n \geq 2$.

∗2 Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, where $1 < p < \infty$. Without using Minkowski's integral inequality or Young's inequality, show that $f * g$ is defined almost everywhere and that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. *Hint*: $|f(x-y)g(y)| = \big(|f(x-y)||g(y)|^p\big)^{1/p} |f(x-y)|^{1-1/p}$.

3 Let $f(x) = 1/\big(1 + x^2(2 + \sin x)\big)$, $x \in \mathbb{R}$. Show that $\widehat{f} \in L^1(\mathbb{R})$.

4 Recall that for $R > 0$, the Fejér kernel F_R is given as an inverse Fourier transform:

$$F_R(x) = \frac{1}{2\pi}\int_{-R}^{R}\left(1 - \frac{|\xi|}{R}\right)e^{ix\xi}\,d\xi = \frac{1 - \cos Rx}{\pi R x^2}, \quad x \in \mathbb{R}.$$

Using these expressions, show that $F_R \in L^1(\mathbb{R})$ and motivate carefully why $\widehat{F_R}(0) = 1$. (This shows that $\int_{\mathbb{R}} F_R(x)\,dx = 1$.)

∗5 Let $f \in L^1(\mathbb{R})$. For $\varepsilon > 0$, show that the formula

$$f_\varepsilon(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\varepsilon|\xi|}\widehat{f}(\xi)e^{ix\xi}\,d\xi$$

defines f_ε for all $x \in \mathbb{R}$, and that $f_\varepsilon \to f$ in $L^1(\mathbb{R})$ as $\varepsilon \to 0$.

6 Show that if $\varphi \in \mathscr{S}(\mathbb{R})$, then $\varphi' \in \mathscr{S}(\mathbb{R})$ and $x\varphi \in \mathscr{S}(\mathbb{R})$, and that if $\varphi_n \to \varphi$ in $\mathscr{S}(\mathbb{R})$, then $\varphi_n' \to \varphi'$ and $x\varphi_n \to x\varphi$ in $\mathscr{S}(\mathbb{R})$. (This shows that the operations $\varphi \mapsto \varphi'$ and $\varphi \mapsto x\varphi$ are continuous on $\mathscr{S}(\mathbb{R})$.)

∗7 Given that $-1 < a < 1$, show that the function

$$\varphi(x) = \frac{e^{ax}}{e^x + e^{-x}}, \quad x \in \mathbb{R},$$

belongs to the Schwartz class $\mathscr{S}(\mathbb{R})$.

8 Let $f(x) = x/(1+x^2)$, $x \in \mathbb{R}$. Show that $f \notin L^1(\mathbb{R})$ but that $f \in L^2(\mathbb{R})$, and determine the Fourier transform $\mathscr{F}f$ by using the sequence $f_n = f\chi_{[-n,n]}$, $n = 1, 2, \ldots$.

9 Let $f(x) = x/(1+x^2)$, $x \in \mathbb{R}$, and let $g(x) = (\sin Rx)/(\pi x)$, $x \neq 0$, where $R > 0$. Show that $f, g \in L^2(\mathbb{R})$, determine $\mathscr{F}f$ and $\mathscr{F}g$ (for instance by looking them up in a table), and calculate $f * g$ by using the formula $f * g = (\mathscr{F}f\,\mathscr{F}g)^{\widehat{\ }-1}$.

∗10 Let $f, g \in L^2(\mathbb{R})$. Show that $2\pi\widehat{fg}(\xi) = (\mathscr{F}f * \mathscr{F}g)(\xi)$ for all $\xi \in \mathbb{R}$ by using sequences φ_n and ψ_n in $\mathscr{S}(\mathbb{R})$ such that $\varphi_n \to f$ and $\psi_n \to g$ in $L^2(\mathbb{R})$.