# TATA66 Exercises, third set

### Hand in solutions to six of the following ten exercises

**∗1** Let $f_n$ and $p_n$ be as in exercise 1 of *TATA66 Exercises, second set*, the results of which may be used freely here.

(a) Show that if $c \geq 0$ is a constant and $g \in C(\mathbb{R})$ satisfies $|g(x) - g(y)| \leq c|x-y|$ for all $x, y \in \mathbb{R}$, then $|g*f_n(x) - g*f_n(y)| \leq c|x-y|$, and $|g*f_n(x) - g(x)| \leq c/2^{n+1}$.

(b) Show that the sequence $p_n$ converges pointwise to a function $p \in \mathscr{D}(\mathbb{R})$.

**2** Consider the following formulas, where $\varphi \in \mathscr{D}(\mathbb{R})$:

(a) $\langle u, \varphi \rangle = \sum_{n=0}^{\infty} 2^n \varphi(n)$, (b) $\langle v, \varphi \rangle = \sum_{n=0}^{\infty} 2^{-n} \varphi^{(n)}(0)$.

Determine whether or not the formulas give well-defined distributions $u, v \in \mathscr{D}'(\mathbb{R})$.

**∗3** Let $D_R$ be the Dirichlet kernel for the line. Show that $D_R \to \delta$ in $\mathscr{D}'(\mathbb{R})$ as $R \to \infty$. *Hint*: Any $\varphi \in \mathscr{D}(\mathbb{R})$ has compact support.

**∗4** Define the function $f$ as follows: $f(x) = n!$ for $2^{-n} \leq x < 2{\cdot}2^{-n}$, where $n = 1, 2, \ldots$ , and $f(x) = 0$ for $x \geq 1$. Show that there does not exist a distribution $u \in \mathscr{D}'(\mathbb{R})$ such that $u = f$ on $]0, \infty[$.

**5** Prove that $\mathrm{pv}(1/x)$ is a well-defined distribution. As a first step, establish that

$$\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}\,dx = -\big(\varphi(\varepsilon) - \varphi(-\varepsilon)\big)\ln\varepsilon - \int_{|x|\geq\varepsilon} \varphi'(x)\ln|x|\,dx, \quad \varphi \in \mathscr{D}(\mathbb{R}),\ \varepsilon > 0.$$

**6** Let $f \in C^\infty(\mathbb{R})$ and $u \in \mathscr{D}'(\mathbb{R})$. Prove, directly from the definitions, that $fu \in \mathscr{D}'(\mathbb{R})$.

**7** Let $f \in C^1(\mathbb{R})$. Show that the function $(f(x) - f(0))/x$, $x \neq 0$, can be extended to a continuous function on $\mathbb{R}$, and that the following equality then holds in $\mathscr{D}'(\mathbb{R})$:

$$f\,\mathrm{pv}(1/x) = f(0)\,\mathrm{pv}(1/x) + \frac{f(x) - f(0)}{x}.$$

(The order of $\mathrm{pv}(1/x)$ is $\leq 1$, so the product $f\,\mathrm{pv}(1/x)$ is well defined.)

**8** Let $u \in \mathscr{D}'(\mathbb{R})$. Define what should be meant by the conjugate $\overline{u}$, the real part $\mathrm{Re}\,u$, and the imaginary part $\mathrm{Im}\,u$ of $u$. Also define what should be meant by saying that $u$ is real. Finally, show that $\delta$ and $\mathrm{pv}(1/x)$ are real, and that $u'$ is real if $u$ is real.

**9** Let $u \in \mathscr{D}'(\mathbb{R})$ be the derivative, in the sense of distributions, of the function $f$ defined by $f(x) = \ln x$, $x > 0$, and $f(x) = 0$, $x \leq 0$. Determine $u(ax)$, where $a > 0$. Is $u$ a homogeneous distribution?

**∗10** Let $H^1(\mathbb{R})$ be the linear subspace of $\mathscr{D}'(\mathbb{R})$ consisting of the $u \in \mathscr{D}'(\mathbb{R})$ for which there exist $u_0, u_1 \in L^2(\mathbb{R})$ such that $u = u_0$ and $u' = u_1$ in $\mathscr{D}'(\mathbb{R})$.

(a) Show that if $u \in H^1(\mathbb{R})$ then $u$ is given by a continuous function.

(b) Show that $H^1(\mathbb{R})$, with inner product given by

$$(u, v) = \int_{\mathbb{R}} u_0(x)\overline{v_0(x)}\,dx + \int_{\mathbb{R}} u_1(x)\overline{v_1(x)}\,dx,$$

is a Hilbert space.