TATA66 Exercises, fourth set

Hand in solutions to six of the following ten exercises

$*$1 Show that the formula $\langle u, \varphi \rangle = \sum_{n=1}^{\infty} (\varphi(2^{-n}) - \varphi(0))$, where $\varphi \in \mathscr{E}(\mathbb{R})$, gives a well-defined element u of $\mathscr{E}'(\mathbb{R})$ and determine the support of u. Also determine if there exist constants C and m such that

$$|\langle u, \varphi \rangle| \le C \sum_{k=0}^{m} \sup_{x \in K} |\varphi^{(k)}(x)|, \quad \varphi \in \mathscr{E}(\mathbb{R}),$$

where K is equal to the support of u.

2 Show that the distribution on $\mathbb{R}$ defined by the function $e^{2x} \sin e^x$ is tempered.

3 Let $u \in \mathscr{D}'(\mathbb{R})$ and $\varphi \in \mathscr{D}(\mathbb{R}^2)$. Show that

$$\left\langle u(x), \int_0^\infty \varphi(x, y)\, dy \right\rangle = \int_0^\infty \big\langle u(x), \varphi(x, y) \big\rangle\, dy.$$

4 Let L be the differential operator $\frac{d^2}{dx^2} - 3\frac{d}{dx} + 2$, and let $f(x) = ae^x + be^{2x}$, $x \ge 0$, and $f(x) = ce^x + de^{2x}$, $x < 0$, where a, b, c, and d are constants.

(a) Find conditions on a, b, c, and d that are equivalent to the equality $Lf = \delta$.

(b) Show that if $Lf = \delta$ and $v \in \mathscr{E}'(\mathbb{R})$, then $u = f * v$ is a solution to the differential equation $Lu = v$.

5 Use the Fourier transform of $\sum_{n=-\infty}^{\infty} \delta_n$ to prove Poisson's summation formula:

$$\sum_{n=-\infty}^{\infty} 2\pi\varphi(2\pi n) = \sum_{n=-\infty}^{\infty} \widehat{\varphi}(n), \quad \varphi \in \mathscr{S}(\mathbb{R}).$$

6 Determine the Fourier transform of the function $u(x) = \arctan x$, $x \in \mathbb{R}$. (Be sure to handle any division problems that arise carefully, as always.)

7 Let H be the linear operator defined on $\mathscr{S}(\mathbb{R})$ by $H\varphi = (1/\pi)\,\mathrm{pv}(1/x) * \varphi$, $\varphi \in \mathscr{S}(\mathbb{R})$. Show that $\|H\varphi\|_2 = \|\varphi\|_2$, $\varphi \in \mathscr{S}(\mathbb{R})$.

$*$8 Let $u_n(x) = (x + \frac{i}{n})^{-1}$, where $x \in \mathbb{R}$ and $n \ge 1$.

(a) Verify that $u_n \in L^2(\mathbb{R})$ and calculate $\widehat{u_n}$.

(b) Prove that $\widehat{u_n} \to -2\pi i\chi$ in $\mathscr{S}'(\mathbb{R})$ as $n \to \infty$. (χ is the step function.)

(c) Use the above to determine $\lim_{n\to\infty} u_n$ in $\mathscr{S}'(\mathbb{R})$.

$*$9 Let $u = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ and for $n \ge 1$ let $u_n = u * u * \ldots * u$ (n factors).

(a) Express u_n as a sum of terms of the form $c\delta_a$ with $c, a \in \mathbb{R}$.

(b) Show that $\widehat{u}(\xi) = \cos\xi$ and that $\widehat{u_n}(\xi) = \cos^n \xi$.

(c) Let $v_n(x) = \sqrt{n}\, u_n(\sqrt{n}\, x)$. Prove that $\widehat{v_n}(\xi) \to e^{-\xi^2/2}$ in $\mathscr{S}'(\mathbb{R})$ as $n \to \infty$.

(d) Use the above to determine $\lim_{n\to\infty} v_n$ in $\mathscr{S}'(\mathbb{R})$.

$*$10 Show that the Maclaurin series $\sum_{k=0}^{\infty} (-1)^k x^{2k}/(2k)!$ for $\cos x$ is convergent in $\mathscr{D}'(\mathbb{R})$ but not in $\mathscr{S}'(\mathbb{R})$.