6.16)

$D: 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 4.$

D har volym $1 \cdot 2 \cdot 4 = 8.$

$$\sqrt[3]{x} + y + \sqrt{z} \le \sqrt[3]{1} + 2 + \sqrt{4} = 5$$

$$I \le 8 \cdot 5 = 40.$$

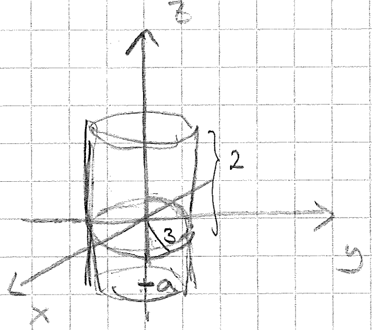
$$\int_0^1 \left(\int_0^2 \left(\int_0^4 (\sqrt[3]{x} + y + \sqrt{z})\, dz \right) dy \right) dx =$$

$$= 8 \int_0^1 \sqrt[3]{x}\, dx + 4 \int_0^2 y\, dy + 2 \int_0^4 \sqrt{z}\, dz =$$

$$= 8 \left[\tfrac{3}{4} x^{4/3} \right]_0^1 + 4 \left[y^2/2 \right]_0^2 + 2 \left[\tfrac{2}{3} z^{3/2} \right]_0^4 =$$

$$= 6 + 8 + \frac{32}{3} = \underline{\underline{\frac{74}{3}}}$$

6.17)



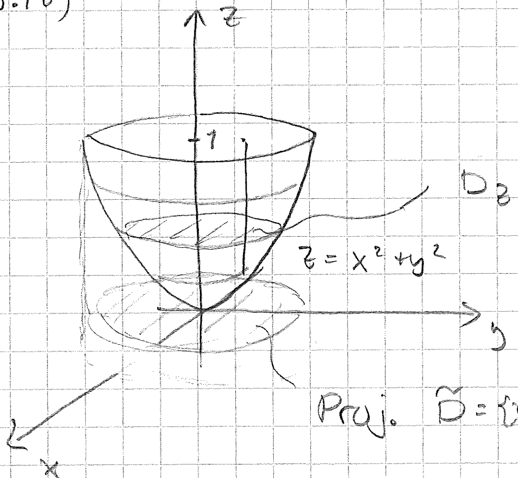
$D = \{(x,y,z) : x^2+y^2 \le 9,\ a \le z \le 2+a\}$

$$\iiint_D (x^2+y^2) z\, dx\,dy\,dz = \iint_{x^2+y^2 \le 9} \left(\int_a^{2+a} (x^2+y^2) z\, dz \right) dx\,dy$$

$$= \int_0^{2\pi}\int_0^3 \rho^2 \rho\, d\rho\, d\varphi \cdot \int_a^{2+a} z\, dz = 2\pi \left[\frac{\rho^4}{4} \right]_0^3 \cdot \left[\frac{z^2}{2} \right]_a^{2+a} =$$

$$= \frac{81\pi}{4} \left((2+a)^2 - a^2 \right) = 81\pi(1+a).$$

6.18)



$D_z = \{(x,y) : x^2+y^2 \leq z\}$, $a=0, b=1$

Proj. $\tilde{D} = \{(x,y) : x^2+y^2 \leq 1\}$

$A(x,y) = x^2+y^2$, $B(x,y)=1$

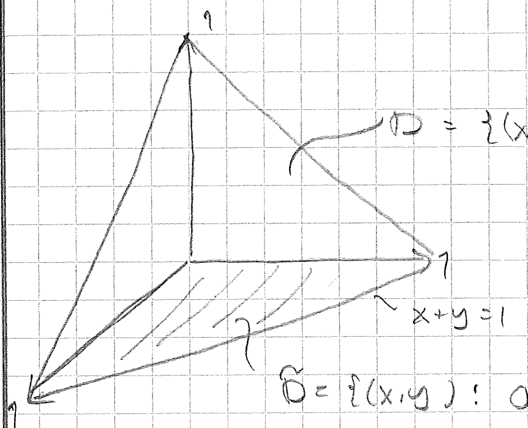
$$\iiint_D z\sqrt{x^2+y^2}\,dx\,dy\,dz = \iint_{\tilde{D}} \left(\int_{x^2+y^2}^{1} z\sqrt{x^2+y^2}\,dz \right) dx\,dy =$$

$$= \iint_{\tilde{D}} \sqrt{x^2+y^2} \left(\frac{1}{2} - \frac{(x^2+y^2)^2}{2} \right) dx\,dy =$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\int_0^1 \rho^2(1-\rho^4)\rho\,d\rho \right) d\varphi =$$

$$= \pi \int_0^1 (\rho^2 - \rho^6)\,d\rho = \pi \left[\frac{\rho^3}{3} - \frac{\rho^7}{7} \right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{7}\right) = \underline{\underline{\frac{4\pi}{21}}}$$

6.19)



$D = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}$

$G = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$

$$\iiint_D \frac{dx\,dy\,dz}{(1+x+y+z)^3} = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} \frac{1}{(1+x+y+z)^3}\, dz \right) dy \right) dx =$$

$$= \int_0^1 \left(\int_0^{1-x} \left[\frac{-1}{2(1+x+y+z)^2} \right]_{z=0}^{1-x-y} dy \right) dx =$$

$$= \int_0^1 \left(\int_0^{1-x} \left(\frac{-1}{2 \cdot 2^2} + \frac{1}{2(1+x+y)^2} \right) dy \right) dx =$$

$$= \int_0^1 \left[\frac{-y}{8} - \frac{1}{2(1+x+y)} \right]_{y=0}^{1-x} dx =$$

$$= \int_0^1 \left(\frac{-(1-x)}{8} - \frac{1}{2 \cdot 2} + \frac{1}{2(1+x)} \right) dx =$$

$$= \left[\frac{x^2}{16} - \frac{3x}{8} + \frac{1}{2} \ln(1+x) \right]_0^1 = \frac{1}{16} - \frac{3}{8} + \frac{1}{2} \ln 2 =$$

$$= \underline{\underline{\ln \sqrt{2} - \frac{5}{16}}}\,.$$

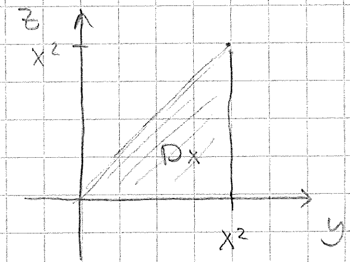
6.20)

$D = \{(x,y,z) : 0 \le z \le y \le x^2 \le 1\}$

a) $D = \{(x,y,z) : 0 \le x^2 \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}$

$= \{(x,y,z) : -1 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}$.

$D_x = \{(y,z) : 0 \le y \le x^2,\ 0 \le z \le y\}$, $-1 \le x \le 1$.



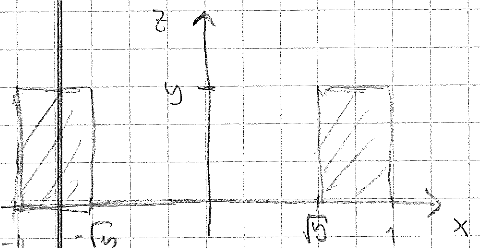
b)

$0 \le y \le 1$,

$D_y = \{(x,z) : 0 \le z \le y,\ y \le x^2 \le 1\} =$

$= \{(x,z) : 0 \le z \le y,\ \sqrt{y} \le x \le 1\}$

$\cup \{(x,z) : 0 \le z \le y,\ \sqrt{y} \le -x \le 1\}$

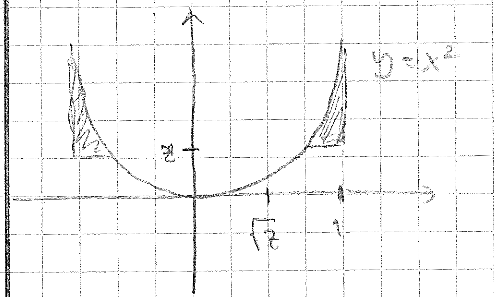


c)

$0 \le z \le 1$

$D_z = \{(x,y) : z \le y \le x^2,\ z \le x^2 \le 1\}$

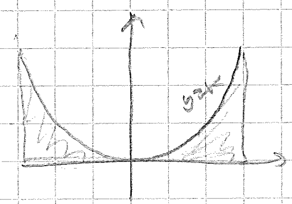
$= \{(x,y) : z \le y \le x^2,\ \sqrt{z} \le x \le 1\} \cup \{(x,y) : z \le y \le x^2,\ \sqrt{z} \le -x \le 1\}$



6.20)

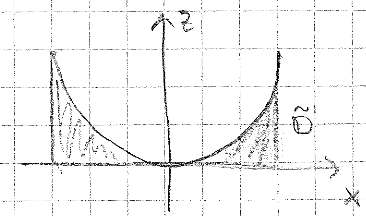
d) $\tilde{D} = \{(x,y) : 0 \le y \le x^2, \ 0 \le x^2 \le 1\} =$

$= \{(x,y) : 0 \le y \le x^2, \ -1 \le x \le 1\}$



$D_{xy} = \{z : 0 \le z \le y\}$

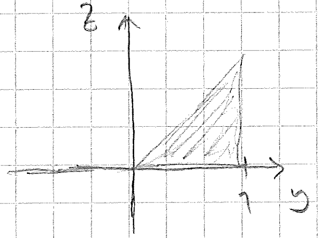
e) $\tilde{D} = \{(x,z) : 0 \le z \le x^2 \le 1\}$



$D_{xz} = \{y : z \le y \le x^2\}$

f) $\tilde{D}_{yz} = \{(y,z) : 0 \le z \le y \le 1\}$, $D_{yz} = \{x : -1 \le x \le -\sqrt{y}\} \cup$

$\cup \{x : \sqrt{y} \le x \le 1\}$



6.21)

$$\iiint_D \frac{z}{1+x^2}\,dx\,dy\,dz =$$

$D = \{(x,y,z):\ 0 \le z \le y \le x^2 \le 1\}$

$\tilde{D} = \{(x,y):\ 0 \le y \le x^2,\ -1 \le x \le 1\}, \quad 0 \le z \le y$

$$= \iint_{\tilde{D}} \left(\int_0^y \frac{z}{1+x^2}\,dz \right) dx\,dy = \iint_{\tilde{D}} \frac{y^2}{2(1+x^2)}\,dx\,dy =$$

$$= \int_{-1}^{1} \left(\int_0^{x^2} \frac{y^2}{2(1+x^2)}\,dy \right) dx = \int_{-1}^{1} \frac{x^6}{6(1+x^2)}\,dx$$

$$\frac{x^6}{6(1+x^2)} = \frac{x^4(1+x^2) - x^4}{6(1+x^2)} = \frac{x^4}{6} - \frac{x^2(1+x^2) - x^2}{6(1+x^2)} =$$

$$= \frac{x^4 - x^2}{6} + \frac{x^2}{6(1+x^2)} = \frac{x^4 - x^2 + 1}{6} - \frac{1}{6(1+x^2)}$$

$$= \left[\frac{x^5}{30} - \frac{x^3}{18} + \frac{x}{6} - \frac{1}{6}\arctan x \right]_{-1}^{1} =$$

$$2\left(\frac{1}{30} - \frac{1}{18} + \frac{1}{6} - \frac{\pi}{24} \right) = \frac{1}{15} - \frac{1}{9} + \frac{1}{3} - \frac{\pi}{12} = \frac{3 - 5 + 15}{45} - \frac{\pi}{12} =$$

$$= \frac{13}{45} - \frac{\pi}{12}$$

6.24)

D begränsas av $x \geq 0, y \geq 0, z \geq 0$, $z = y$ och $z = 4 - x^2$

Projektion $\tilde{D}$ på xz-planet ges av

$0 \leq x \leq 2$, $0 \leq z \leq 4 - x^2$, och för varje

$(x, z) \in \tilde{D}$ får vi gränserna $0 \leq y \leq z$.

$$\iiint_D 2x\,dx\,dy\,dz = \iint_{\tilde{D}} \left(\int_0^z 2x\,dy \right) dx\,dz =$$

$$= \int_0^2 \left(\int_0^{4-x^2} 2xz\,dz \right) dx = \int_0^2 \left[xz^2 \right]_{z=0}^{4-x^2} dx = \int_0^2 x(4-x^2)^2\,dx =$$

$$= \left[\frac{-(4-x^2)^3}{3 \cdot 2} \right]_0^2 = \frac{32}{3}$$

6.25)

$$I = \iiint_D 3z\,dx\,dy\,dz, \qquad D \text{ har volym } \frac{2\pi}{3}.$$

Eftersom $0 \leq 3z \leq 3$ på D gäller att

$$0 \leq I \leq 3 \cdot \frac{2\pi}{3} = 2\pi.$$

D ges i polära koordinater av $0 \leq r \leq 1$, $0 \leq \varphi \leq 2\pi$, $0 \leq \theta \leq \pi/2$

$$I = \int_0^{\pi/2} \left(\int_0^{2\pi} \left(\int_0^1 3r\cos\theta\, r^2 \sin\theta\, dr \right) d\varphi \right) d\theta =$$

$$= 2\pi \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta \cdot \int_0^1 3r^3\,dr = 2\pi \int_0^{\pi/2} \frac{\sin 2\theta}{2}\, d\theta \cdot \left[\frac{3r^4}{4} \right]_0^1 =$$

$$= \frac{3\pi}{2} \left[-\frac{\cos 2\theta}{4} \right]_0^{\pi/2} = \underline{\underline{\frac{3\pi}{4}}}$$

6.26)

a)

$$\iiint_{B(0,1)} (z^2-x^2-y^2)\,dx\,dy\,dz = \int_0^{\pi}\left(\int_0^{2\pi}\left(\int_0^1 (r^2\cos^2\theta - r^2\sin^2\theta)\,r^2\sin\theta\,dr\right)d\varphi\right)d\theta =$$

$$= 2\pi\int_0^{\pi}(2\cos^2\theta - 1)\sin\theta\,d\theta \cdot \int_0^1 r^4\,dr =$$

$$= 2\pi\left[\cos\theta - \frac{2\cos^3\theta}{3}\right]_0^{\pi} \cdot \left[\frac{r^5}{5}\right]_0^1 = -\frac{4\pi}{15}$$

b) Gränser i sfäriska koordinater $0 \le r \le 1$, $-\pi/2 \le \varphi \le \pi/2$, $0 \le \theta \le \pi$

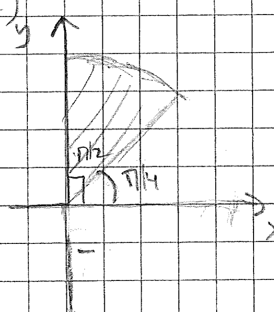
$$\int_0^{\pi}\left(\int_{-\pi/2}^{\pi/2}\left(\int_0^1 r\cos\varphi\sin\theta\, e^{r^2}\cdot r^2\sin\theta\,dr\right)d\varphi\right)d\theta =$$

$$= \int_0^{\pi}\sin^2\theta\,d\theta \cdot \int_{-\pi/2}^{\pi/2}\cos\varphi\,d\varphi \cdot \int_0^1 r^3 e^{r^2}\,dr =$$

$$= \int_0^{\pi}\frac{1-\cos 2\theta}{2}\,d\theta \cdot \left[\sin\varphi\right]_{-\pi/2}^{\pi/2}\left(\left[r^2\frac{e^{r^2}}{2}\right]_0^1 - \int_0^1 r e^{r^2}\,dr\right) =$$

$$= \frac{\pi}{2}\cdot 2\cdot\left[r^2\frac{e^{r^2}}{2} - \frac{e^{r^2}}{2}\right]_0^1 = \pi\left(\frac{e}{2} - \frac{e}{2} - 0 + \frac{1}{2}\right) = \frac{\pi}{2}$$

c)



$0 \le r \le \sqrt{3}$, $\pi/4 \le \varphi \le \pi/2$, $0 \le \theta \le \pi$.

$$\int_0^{\pi}\left(\int_{\pi/4}^{\pi/2}\left(\int_0^{\sqrt{3}} r\cos\varphi\sin\theta\, r^2\sin\theta\,dr\right)d\varphi\right)d\theta =$$

$$= \int_0^{\pi}\sin^2\theta\,d\theta\int_{\pi/4}^{\pi/2}\cos\varphi\,d\varphi\int_0^{\sqrt{3}} r^3\,dr = \frac{\pi}{2}\cdot\left[\sin\varphi\right]_{\pi/4}^{\pi/2}\cdot\left[\frac{r^4}{4}\right]_0^{\sqrt{3}} =$$

$$= \frac{\pi}{2}\cdot\left(1 - \frac{1}{\sqrt{2}}\right)\cdot\frac{9}{4} = \frac{9\pi(2-\sqrt{2})}{16}$$

6.27)

a)
$$\begin{cases} u = x+y+z \\ v = x+2y+3z \\ w = x+4y+9z \end{cases} \Leftrightarrow \dots \Leftrightarrow \begin{cases} x = 3u - \frac{5v}{2} + \frac{w}{2} \\ y = -3u+4v-w \\ z = \frac{2u-3v+w}{2} \end{cases}$$

$$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} = 2$$

Så $dxdydz = \frac{1}{2} dudvdw$

$0 \le u \le 1, \quad 0 \le v \le 1, \quad 0 \le w \le 1$

$y - x - z = \dots = -7u + 8v - 2w$

$$\iiint_D (y-x-z)\,dxdydz = \int_0^1 \left(\int_0^1 \left(\int_0^1 (-7u+8v-2w)\frac{1}{2}\,du \right) dv \right) dw$$

$$= /\text{Av symmetriskäl}/ = -\frac{1}{2}\int_0^1 u\,du = \underline{\underline{-\frac{1}{4}}}$$

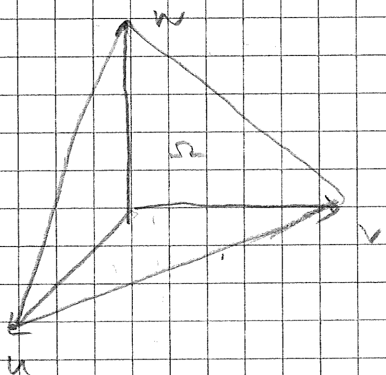
b)
Med basbyte till basen

$((1,1,0),(1,0,1),(0,1,1))$ får vi

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} \Leftrightarrow \begin{cases} x = u+v \\ y = u+w \\ z = v+w \end{cases} \qquad \frac{d(x,y,z)}{d(u,v,w)} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2$$

Punkterna $(0,0,0), (1,1,0), (1,0,1)$ och $(0,1,1)$

får i uvw koordinaterna $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$.



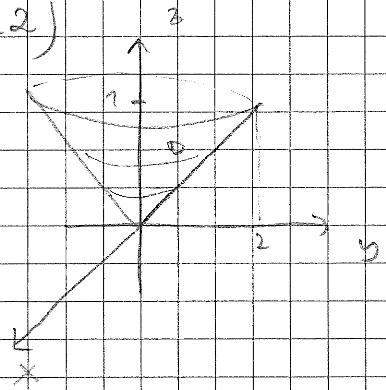
$\Omega: \; 0 \le u \le 1, \quad 0 \le v \le 1-u, \quad 0 \le w \le 1-u-v.$

$$\iiint_D x\,dxdydz = \iiint_\Omega (u+v)\,2\,dudvdw$$

$$= /\text{symm}/ = 4 \iiint_\Omega u\,dudvdw =$$

$$= 4\int_0^1 \left(\int_0^{1-u} \left(\int_0^{1-u-v} u\,dw \right) dv \right) du = 4\int_0^1 \left(\int_0^{1-u} u(1-u-v)\,dv \right) du = \dots = \frac{1}{6}$$

6.22)



D: $0 \le z \le 1$, $\sqrt{x^2+y^2} \le 2z$

$$\int_0^1 \left(\int_0^{2\pi} \left(\int_0^{2z} (\rho^2 \sin^2\varphi + z^2)\rho\, d\rho \right) d\varphi \right) dz =$$

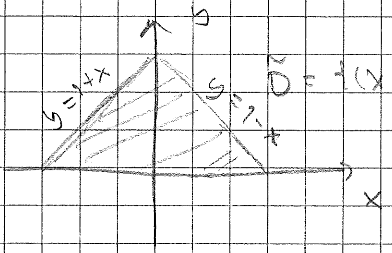
$$= \int_0^1 \left(\int_0^{2\pi} \left[\frac{\rho^4}{4}\sin^2\varphi + \frac{\rho^2}{2} z^2 \right]_{\rho=0}^{2z} d\varphi \right) dz = \int_0^1 \left(\int_0^{2\pi} (4z^4\sin^2\varphi + 2z^4)\, d\varphi \right) dz$$

$$= \int_0^1 2z^4\, dz \cdot \int_0^{2\pi} (1+2\sin^2\varphi)\, d\varphi = \left[\frac{2z^5}{5}\right]_0^1 \cdot \left[\varphi + \varphi - \frac{\sin 2\varphi}{2}\right]_0^{2\pi}$$

$$= \underline{\underline{\frac{8\pi}{5}}}$$

6.23) $x-y \le z \le x+y$, $|x|+|y| \le 1$.

$x-y \le x+y \iff y \ge 0$



$\tilde{D} = \{(x,y) : y \ge 0, |x|+|y| \le 1\} =$

$= \{(x,y) : 0 \le y \le 1, \; y-1 \le x \le 1-y\}$

$$\iiint_D y e^z \, dx\,dy\,dz = \iint_{\tilde{D}} \left(\int_{x-y}^{x+y} y e^z \, dz \right) dx\,dy =$$

$$= \iint_{\tilde{D}} y\left(e^{x+y} - e^{x-y}\right) dx\,dy = \int_0^1 \left(\int_{y-1}^{1-y} y\left(e^{y} - e^{-y}\right) e^x \, dx \right) dy =$$

$$= \int_0^1 y\left(e^{y} - e^{-y}\right) \cdot \left(e^{1-y} - e^{y-1}\right) dy = \dots = \underline{\underline{\frac{1}{e}}}$$