2.19) $z'_x + z'_y = 0$

a) $z = \sin(x-y)$, $z'_x = \cos(x-y)$, $z'_y = -\cos(x-y)$

så $z'_x + z'_y = \cos(x-y) - \cos(x-y) = 0$

$z = 1 + (x-y)e^{-x}e^{y}$, $z'_x = e^{-x}e^{y} - (x-y)e^{-x}e^{y}$

$z'_y = -e^{-x}e^{y} + (x-y)e^{-x}e^{y}$

$z'_x + z'_y = e^{-x}e^{y} - (x-y)e^{-x}e^{y} - e^{-x}e^{y} + (x-y)e^{-x}e^{y} = 0.$

b) $z(x,y) = f(x-y)$, $z'_x = f'(x-y)$, $z'_y = -f(x-y)$

$z'_x + z'_y = f(x-y) - f(x-y) = 0.$

c) $\sin(x-y) = f(x-y)$ där $f(t) = \sin t$

$1 + (x-y)e^{-x}e^{y} = 1 + (x-y)e^{-(x-y)} = f(x-y)$ där

$f(t) = 1 + te^{-t}.$

2.20) $z = f(x/y)$, $z'_x = f'\left(\frac{x}{y}\right)\cdot\frac{1}{y}$, $z'_y = f'\left(\frac{x}{y}\right)\cdot\left(-\frac{x}{y^2}\right)$

$x z'_x + y z'_y = \frac{x}{y} f'\left(\frac{x}{y}\right) - \frac{xy}{y^2} f'\left(\frac{x}{y}\right) = 0.$

$\frac{x^2-y^2}{xy} = \frac{x}{y} - \frac{y}{x} = f\left(\frac{x}{y}\right)$

där $f(t) = t - \frac{1}{t}.$

2.21) $z'_x = z'_u u'_x + z'_v v'_x$

$z'_y = z'_u u'_y + z'_v v'_y.$

a) $\begin{cases} u = x+y \\ v = xy \end{cases}$ ger $\begin{matrix} u'_x = 1, u'_y = 1 \\ v'_x = y, v'_y = x \end{matrix}$

så $z'_x = z'_u + y z'_v$, $z'_y = z'_u + x z'_v$

b) $\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$ ger $u'_x = 2x$, $u'_y = -2y$, $v'_x = 2y$, $v'_y = 2x$

så $z'_x = 2x z'_u + 2y z'_v$,

$z'_y = -2y z'_u + 2x z'_v$

c) $\begin{cases} u = 2xy \\ v = 1/y \end{cases}$ ger $u'_x = 2y$, $u'_y = 2x$, $v'_x = 0$, $v'_y = -1/y^2$

så $z'_x = 2y z'_u$, $z'_y = 2x z'_u - \frac{1}{y^2} z'_v.$

2.22)

a) $\begin{cases} u = x-y \\ v = x+y \end{cases}$ $u'_x = 1,\ u'_y = -1,\ v'_x = 1,\ v'_y = 1$

$z'_x = z'_u u'_x + z'_v v'_x = z'_u + z'_v$

$z'_y = z'_u u'_y + z'_v v'_y = -z'_u + z'_v$

$z'_x + z'_y = z'_u + z'_v - z'_u + z'_v = 2z'_v = 0$

$\Leftrightarrow \quad z'_v = 0 \quad \Leftrightarrow \quad \underline{\underline{z = f(u) = f(x-y)}}$

b)

$z(0,y) = f(-y) = y - \cos y$

$\Leftrightarrow \quad f(y) = (-y) - \cos(-y) = -y - \cos y$

$z = f(x-y) = -(x-y) - \cos(x-y) = \underline{\underline{y - x - \cos(x-y)}}$

2.23) $\begin{cases} u = 2x-3y \\ v = x \end{cases} \Leftrightarrow \begin{cases} x = v \\ y = \frac{2v-u}{3} \end{cases}$

a) $\frac{\partial u}{\partial x} = 2 \quad , \frac{\partial x}{\partial u} = 0 . \qquad \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} = 0 \neq 1.$

b) $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 2\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$.

När vi beräknar $\frac{\partial f}{\partial v}$ hålls u konstant,

när vi beräknar $\frac{\partial f}{\partial x}$ hålls y konstant.

2.26)

$$2x z'_x - y z'_y = y + xy \qquad x>0,\ y>0.$$

$$\begin{cases} u = xy^2 & u'_x = y^2,\ u'_y = 2xy \\ v = y & v'_x = 0,\ v'_y = 1 \end{cases}$$

$$z'_x = z'_u u'_x + z'_v v'_x = y^2 z'_u$$

$$z'_y = z'_u u'_y + z'_v v'_y = 2xy z'_u + z'_v$$

$$2x z'_x - y z'_y = 2xy^2 z'_u - 2xy^2 z'_u - y z'_v = y + xy$$

$$\Leftrightarrow \quad z'_v = -\frac{y+xy}{y} = -1 - x = -1 - u/v^2$$

$$\Rightarrow \quad z = -v + \frac{u}{v} + g(u).$$

D.v.s. $\underline{\underline{z = -y + xy + g(xy^2)}}$

$$z(1,y) = -y + y + g(y^2) = e^{-y} = e^{-\sqrt{y^2}}$$

Så $g(t) = e^{-\sqrt{t}}$, d.v.s.

$$\underline{\underline{z = -y + xy + e^{-\sqrt{xy^2}}}}.$$

2.31)

$$\frac{dw}{dx} = \frac{\partial w}{\partial u}\frac{du}{dx} + \frac{\partial w}{\partial v}\frac{dv}{dx} = \frac{1}{x}\frac{\partial w}{\partial u} + 2x\frac{\partial w}{\partial v} \qquad (*)$$

gäller för alla w.

Med $w = z$ i $(*)$ får vi

$$\frac{dz}{dx} = \frac{1}{x}\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}.$$

$$\frac{d^2z}{dx^2} = \frac{d}{dx}\left(\frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{\partial z}{\partial u} + 2x\frac{\partial z}{\partial v}\right) =$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\frac{d}{dx}\left(\frac{\partial z}{\partial u}\right) + 2\frac{\partial z}{\partial v} + 2x\frac{d}{dx}\left(\frac{\partial z}{\partial v}\right)$$

$$= \Big/\ w = \frac{\partial z}{\partial u} \text{ resp. } w = \frac{\partial z}{\partial v} \text{ i } (*)\ \Big/ =$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\left(\frac{1}{x}\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) + 2x\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\right) + 2\frac{\partial z}{\partial v} +$$

$$+ 2x\left(\frac{1}{x}\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) + 2x\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\right) =$$

$$= \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} + 4\frac{\partial^2 z}{\partial u \partial v} + 4x^2\frac{\partial^2 z}{\partial v^2} - \frac{1}{x^2}\frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial v}.$$

2.32)

$$z''_{xx} - 4z''_{xy} + 4z''_{yy} = 6y$$

a) $\begin{cases} u = 2x + y \\ v = x \end{cases} \Leftrightarrow \begin{cases} x = v \\ y = u - 2v \end{cases}$

$$w'_x = w'_u u'_x + w'_v v'_x = 2w'_u + w'_v \qquad (*)$$

$$w'_y = w'_u u'_y + w'_v v'_y = w'_u \qquad (**)$$

$$z''_{xx} = (z'_x)'_x = /w = z \text{ i } (*)/ = (2z'_u + z'_v)'_x =$$

$$= 2(z'_u)'_x + (z'_v)'_x = /w = z'_u \text{ resp. } w = z'_v/ =$$

$$= 2\left(2(z'_u)'_u + (z'_u)'_v\right) + \left(2(z'_v)'_u + (z'_v)'_v\right) =$$

$$= 4z''_{uu} + 4z''_{uv} + z''_{vv}.$$

$$z''_{yy} = (z'_y)'_y = /w = z \text{ i } (**)/ = (z'_u)'_y =$$

$$/w = z'_u \text{ i } (**)/ = (z'_u)'_u = z''_{uu}$$

$$z''_{xy} = (z'_x)'_y = /w = z \text{ i } (*)/ = (2z'_u + z'_v)'_y =$$

$$= 2(z'_u)'_y + (z'_v)'_y = /w = z'_u \text{ resp. } w = z'_v \text{ i } (**)/$$

$$= 2(z'_u)'_u + (z'_v)'_u = 2z''_{uu} + z''_{uv}.$$

$$z''_{xx} - 4z''_{xy} + 4z''_{yy} = 4z''_{uu} + 4z''_{uv} + z''_{vv} - 8z''_{uu} - 4z''_{uv} + 4z''_{uu}$$

$$= z''_{vv} = 6y = 6(u - 2v) = 6u - 12v$$

$$\Rightarrow z'_v = 6uv - 6v^2 + h(u) \Rightarrow z = 3uv^2 - 2v^3 + vh(u) + k(u)$$

$$= 3(2x+y)x^2 - 2x^3 + xh(2x+y) + k(2x+y) =$$

$$= 4x^3 + 3x^2y + xh(2x+y) + k(2x+y)$$

$$h, k \in C^2.$$

2.32 b) $z(0,y) = k(y) = e^{-y^2}$.

c) $z(0,y) = k(y) = e^{-y^2}$

$z'_x = 12x^2 + 6xy + h(2x+y) + 2x\,h'(2x+y) + 2k'(2x+y)$

$z'_x(0,y) = h(y) + 2k'(y) = h(y) - 4ye^{-y^2} = 0$

$k(y) = e^{-y^2}, \quad h(y) = 4ye^{-y^2}$

Så

$$\underline{\underline{z(x,y) = 4x^3 + 3x^2y + 4x(2x+y)e^{-(2x+y)^2} + e^{-(2x+y)^2}}}$$

d)

$z(0,y) = k(y) = 0$

$z'_x = 12x^2 + 6xy + h(2x+y) + 2x\,h'(2x+y) + 2k'(2x+y)$

Med $k = 0$, $z'_x(x,-8x) = 0$ får vi

$z'_x(x,-8x) = 12x^2 + 6x(-8x) + h(-6x) + 2x\,h'(-6x) = 0$

$\Leftrightarrow \quad 2x\,h'(-6x) + h(-6x) = 36x^2$

$\Leftrightarrow \quad h'(-6x) + \frac{1}{2x}h(-6x) = \frac{36x^2}{2x} = 18x$

$t = -6x \Leftrightarrow x = -t/6$ ger

$h'(t) - \frac{3}{t}h(t) = -3t$

$\int -\frac{3}{t}\,dt = -3\ln t \qquad$ I.F. $\quad \frac{1}{t^3}$

$\left(\frac{1}{t^3}h(t)\right)' = -\frac{3}{t^2}$

$\Leftrightarrow \quad \frac{1}{t^3}h(t) = \frac{3}{t} + C$

$\Leftrightarrow \quad h(t) = 3t^2 + Ct^3$.

$z(x,y) = 4x^3 + 3x^2y + x\left(3(2x+y)^2 + C(2x+y)^3\right) =$

$= 16x^3 + 15x^2y + 3xy^2 + Cx(2x+y)^3$

OBS! Om vi ser på ekvationen

$2x\,h'(-6x) + h(-6x) = -\frac{t}{3}h'(t) + h(t) = 36x^2 = t^2$,

Om vi skarvar ihop lösningar $h(t) = \begin{cases} 3t^2 + C_1t^3 & t \geq 0 \\ 3t^2 + C_2t^3 & t < 0 \end{cases}$

så blir ändå en lösning. Men antar vi $h \in C^3$ måste $C_1 = C_2$ ...

2.33)

$$\begin{cases} u = x + y \\ v = xy \end{cases}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right) =$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial z}{\partial v} =$$

$$= \Big/ \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v} \Big/ =$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) + y\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right) + x\left(\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) + y\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\right) + \frac{\partial z}{\partial v} =$$

$$= \frac{\partial^2 z}{\partial u^2} + (x+y)\frac{\partial^2 z}{\partial u \partial v} + xy\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v} =$$

$$= \frac{\partial^2 z}{\partial u^2} + u\frac{\partial^2 z}{\partial u \partial v} + v\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}.$$

2.34)

$$x\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial z}{\partial x} = x \qquad x>0,\ y>0.$$

$$\begin{cases} u = 2xy \\ v = 1/y \end{cases} .$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = 2y\frac{\partial w}{\partial u} \qquad (*)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = 2x\frac{\partial w}{\partial u} - \frac{1}{y^2}\frac{\partial w}{\partial v} \qquad (**).$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \Big/ w = z \text{ i } (*) \Big/ =$$

$$= \frac{\partial}{\partial x}\left(2y\frac{\partial z}{\partial u}\right) = 2y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) = \Big/ w = \frac{\partial z}{\partial u} \text{ i } (*) \Big/$$

$$= 2y\left(2y\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\right) = 4y^2\frac{\partial^2 z}{\partial u^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \Big/ w = z \text{ i } (**) \Big/ =$$

$$= \frac{\partial}{\partial x}\left(2x\frac{\partial z}{\partial u} - \frac{1}{y^2}\frac{\partial z}{\partial v}\right) = 2\frac{\partial z}{\partial u} + 2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) +$$

$$- \frac{1}{y^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = \Big/ w = \frac{\partial z}{\partial u} \text{ resp. } w = \frac{\partial z}{\partial v} \text{ i } (*) \Big/$$

$$= 2\frac{\partial z}{\partial u} + 4xy\frac{\partial^2 z}{\partial u^2} - \frac{2}{y}\frac{\partial^2 z}{\partial u\partial v}$$

$$x\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial z}{\partial x} = \Big/ w = z \text{ i } (*) \Big/ =$$

$$= x\left(4y^2\frac{\partial^2 z}{\partial u^2}\right) - y\left(2\frac{\partial z}{\partial u} + 4xy\frac{\partial^2 z}{\partial u^2} - \frac{2}{y}\frac{\partial^2 z}{\partial u\partial v}\right) + 2y\frac{\partial z}{\partial u} =$$

$$= 2\frac{\partial^2 z}{\partial u\partial v} = x = \frac{uv}{2} \iff z''_{uv} = \frac{uv}{4} \Rightarrow z'_u = \frac{uv^2}{8} + h(u)$$

$$\Rightarrow z = \frac{u^2v^2}{16} + H(u) + K(v) \qquad H, K \in C^2.$$

$$\iff z = \frac{x^2}{4} + H(2xy) + K(1/y) = \underline{\underline{\frac{x^2}{4} + f(xy) + g(y).}}$$

2.25)

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial T}{\partial \rho}$$

$$T(\rho, t) = f(\rho^2/t)$$

$$\frac{\partial T}{\partial \rho} = f'(\rho^2/t)\frac{2\rho}{t}, \quad \frac{\partial^2 T}{\partial \rho^2} = f''(\rho^2/t)\frac{4\rho^2}{t^2} + \frac{2}{t} f'(\rho^2/t),$$

$$\frac{\partial T}{\partial t} = f'(\rho^2/t)\left(-\frac{\rho^2}{t^2}\right)$$

$$f'(\rho^2/t)\left(-\frac{\rho^2}{t^2}\right) = f''(\rho^2/t)\frac{4\rho^2}{t^2} + \frac{2}{t} f'(\rho^2/t) - \frac{2}{t} f'(\rho^2/t)$$

$$\Leftrightarrow \quad 4 f''(\rho^2/t)\frac{\rho^2}{t^2} + f'(\rho^2/t)\frac{\rho^2}{t^2} = 0$$

$$\Leftrightarrow \quad f'' + \frac{1}{4} f' = 0$$

$$r^2 + \frac{r}{4} = 0 \quad \Leftrightarrow \quad r = 0,\ r = -\frac{1}{4}.$$

$$f(s) = C + De^{-s/4}$$

$$\underline{\underline{T = f(\rho^2/t) = C + De^{-\rho^2/4t}}}.$$

2.27)

$$x f'_x + y f'_y = -f, \quad \begin{cases} u = y/x \\ v = x \end{cases}$$

$$f'_x = f'_u u'_x + f'_v v'_x = -\frac{y}{x^2} f'_u + f'_v$$

$$f'_y = f'_u u'_y + f'_v v'_y = \frac{1}{x} f'_u$$

$$x f'_x + y f'_y = -\frac{y}{x} f'_u + x f'_v + \frac{y}{x} f'_u = x f'_v = v f'_v = -f$$

$$\Leftrightarrow \quad \frac{\partial}{\partial v}(vf) = 0 \quad \Leftrightarrow \quad vf = h(u) \quad \Leftrightarrow \quad f = \frac{h(u)}{v} = \underline{\underline{\frac{h(y/x)}{x}}}.$$

2.28)

$$\begin{cases} g(t) = f(2t, t) \\ h(t) = f(t, -t) \end{cases}$$

$$g'(t) = \frac{\partial f(2t,t)}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \left/ \begin{matrix} x = 2t \\ y = t \end{matrix} \right/ =$$

$$= 2f'_x + f'_y$$

$$h'(t) = \frac{\partial f(t,-t)}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \left/ \begin{matrix} x = t \\ y = -t \end{matrix} \right/ =$$

$$= f'_x - f'_y$$

$$\begin{cases} g'(0) = a = 2f'_x(0,0) + f'_y(0,0) \\ h'(0) = b = f'_x(0,0) - f'_y(0,0) \end{cases} \Leftrightarrow \begin{cases} 3f'_x(0,0) = a+b \\ 3f'_y(0,0) = a-2b \end{cases}$$

SVAR: $f'_x(0,0) = \frac{a+b}{3}$, $f'_y(0,0) = \frac{a-2b}{3}$.

2.29)

$$(*) \quad a(x,y)z'_x + b(x,y)z'_y = 0.$$

a)

$$(x'(t), y'(t)) = c(t)\big(a(x(t),y(t)), b(x(t),y(t))\big)$$

$$\frac{dz(x(t),y(t))}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} =$$

$$= c(t)a(x(t),y(t))\frac{\partial z}{\partial x} + c(t)b(x(t),y(t))\frac{\partial z}{\partial y} =$$

$$= c(t)\left(a(x(t),y(t))\frac{\partial z}{\partial x} + b(x(t),y(t))\frac{\partial z}{\partial y}\right) = 0$$

b) $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$ Antag att $f(x,y) = C$ för en lösning till denna. $\begin{cases} u = f(x,y) \\ v = x \end{cases}$ ger

$z'_x = z'_u u'_x + z'_v v'_x = z'_u f'_x + z'_v$, $z'_y = z'_u u'_y + z'_v v'_y = z'_u f'_y$

$a z'_x + b z'_y = a z'_u f'_x + a z'_v + b z'_u f'_y = z'_u(a f'_x + b f'_y) + a z'_v$

$= a z'_v = 0$, eftersom $\frac{d}{dx} f(x, y(x)) = f'_x + y'(x) f'_y = f'_x + \frac{b}{a} f'_y = 0$.

2.29 c)

$$a z'_x + b z'_y = a z'_v = c(x,y)$$

2.22: $z'_x + z'_y = 0$ $\quad a = b = 1$

$$\frac{dy}{dx} = 1 \iff y = x + c \iff y - x = c$$

$$f(x,y) = y - x.$$

2.26:

$a = 2x,\ b = -y$ $\quad \dfrac{dy}{dx} = -\dfrac{y}{2x}$

$$\int \frac{2}{y}\,dy = \int -\frac{1}{2x}\,dx \iff \ln y^2 = -\ln x + c$$

$$\iff \ln xy^2 = c \iff xy^2 = e^c$$

$$f(x,y) = xy^2$$

2.29 d) $(1-2y)z'_x + (1+2x)z'_y = 0.$

$a = 1-2y,\quad b = 1+2x$

$$\frac{dy}{dx} = \frac{1+2x}{1-2y} \iff \int (1-2y)\,dy = \int (1+2x)\,dx$$

$$\iff y - y^2 = x + x^2 + c$$

$$f(x,y) = x + x^2 - y + y^2 = -c.$$

$$\begin{cases} u = x + x^2 - y + y^2 \\ v = x \end{cases}$$

ger $a z'_v = (1-2y) z'_v = 0 \iff z'_v = 0$ (ω $y = 1/2$).

$$z = h(u) = h(x + x^2 - y + y^2) \qquad h \in C^1.$$

2.3 6) $\begin{cases} u = \cos\phi\, x + \sin\phi\, y \\ v = -\sin\phi\, x + \cos\phi\, y. \end{cases}$ OBS! ϕ konstant.

$$w'_x = w'_u u'_x + w'_v v'_x = \cos\phi\, w'_u - \sin\phi\, w'_v$$

$$w'_y = w'_u u'_y + w'_v v'_y = +\sin\phi\, w'_u + \cos\phi\, w'_v$$

$$z''_{xx} = (z'_x)'_x = (\cos\phi\, z'_u - \sin\phi\, z'_v)'_x =$$

$$= \cos\phi\,(z'_u)'_x - \sin\phi\,(z'_v)'_x =$$

$$= \cos\phi\,(\cos\phi\, z''_{uu} - \sin\phi\, z''_{uv}) - \sin\phi\,(\cos\phi\, z''_{uv} + \sin\phi\, z''_{vv})$$

$$= \cos^2\phi\, z''_{uu} - 2\cos\phi\sin\phi\, z''_{uv} + \sin^2\phi\, z''_{vv}$$

$$z''_{yy} = (z'_y)'_y = (+\sin\phi\, z'_u + \cos\phi\, z'_v)'_y =$$

$$= +\sin\phi\,(z'_u)'_y + \cos\phi\,(z'_v)'_y =$$

$$= +\sin\phi\,(+\sin\phi\, z''_{uu} + \cos\phi\, z''_{uv}) + \cos\phi\,(+\sin\phi\, z''_{uv} + \cos\phi\, z''_{vv})$$

$$= \sin^2\phi\, z''_{uu} + 2\cos\phi\sin\phi\, z''_{uv} + \cos^2\phi\, z''_{vv}$$

Så $z''_{xx} + z''_{yy} = (\cos^2\phi + \sin^2\phi)\, z''_{uu} + (\cos^2\phi + \sin^2\phi)\, z''_{vv}$

$$= z''_{uu} + z''_{vv}.$$

2.37)

a) $$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho} \qquad \begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$

$$= \cos\varphi\frac{\partial u}{\partial x} + \sin\varphi\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi} = -\rho\sin\varphi\frac{\partial u}{\partial x} + \rho\cos\varphi\frac{\partial u}{\partial y}$$

b) $$\begin{cases} \rho\sin\varphi\frac{\partial u}{\partial \rho} = \rho\sin\varphi\cos\varphi\frac{\partial u}{\partial x} + \rho\sin^2\varphi\frac{\partial u}{\partial y} & (1) \\ \cos\varphi\frac{\partial u}{\partial \varphi} = -\rho\sin\varphi\cos\varphi\frac{\partial u}{\partial x} + \rho\cos^2\varphi\frac{\partial u}{\partial y} & (2) \end{cases}$$

$$(1)+(2) = \rho\sin\varphi\frac{\partial u}{\partial \rho} + \cos\varphi\frac{\partial u}{\partial \varphi} = \rho\frac{\partial u}{\partial y}$$

D.v.s. $$\frac{\partial u}{\partial y} = \sin\varphi\frac{\partial u}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial u}{\partial \varphi}.$$

$$\frac{\partial u}{\partial x} = \ldots = \cos\varphi\frac{\partial u}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial u}{\partial \varphi}.$$

c) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) =$$

$$= \cos\varphi\frac{\partial}{\partial \rho}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\left(\frac{\partial u}{\partial x}\right) + \sin\varphi\frac{\partial}{\partial \rho}\left(\frac{\partial u}{\partial y}\right) + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\left(\frac{\partial u}{\partial y}\right) =$$

$$= \cos\varphi\frac{\partial}{\partial \rho}\left(\cos\varphi\frac{\partial u}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial u}{\partial \varphi}\right) - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\left(\cos\varphi\frac{\partial u}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial u}{\partial \varphi}\right)$$

$$+ \sin\varphi\frac{\partial}{\partial \rho}\left(\sin\varphi\frac{\partial u}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial u}{\partial \varphi}\right) + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial u}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial u}{\partial \varphi}\right)$$

$$= \ldots = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} = 0.$$

d) $$u'' + \frac{1}{\rho}u' = 0 \Leftrightarrow (\rho u')' = 0 \Leftrightarrow \rho u' = 2C \Leftrightarrow u = 2C\ln\rho + D$$

$$= \underline{C\ln(x^2+y^2) + D}.$$

2.39)

$u = xy^2 \quad , \quad z = 2x+y$

Med $\begin{cases} v = x \\ w = 2x+y \end{cases} \iff \begin{cases} x = v \\ y = w - 2v \end{cases}$

skulle

$\left(\frac{\partial u}{\partial x}\right)_y$ var "vanliga" $\frac{\partial u}{\partial x}$, och

$\left(\frac{\partial u}{\partial x}\right)_z = \frac{\partial u}{\partial v}$

D.v.s $\left(\frac{\partial u}{\partial x}\right)_y = \frac{\partial}{\partial x}(xy^2) = \underline{y^2}.$

$\left(\frac{\partial u}{\partial x}\right)_z = \frac{\partial u}{\partial v} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}$

$= \frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} = y^2 - 2(2xy) = \underline{\underline{y^2 - 4xy}}$

2-40)

$$E = f(T, p)$$

$$\left(\frac{\partial E}{\partial T}\right)_p = \frac{\partial f}{\partial T}$$

$$\left(\frac{\partial E}{\partial T}\right)_V = \frac{\partial}{\partial T} f(T, p(T,V)) = \frac{\partial f}{\partial T} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial T}$$

$$\left(\frac{\partial E}{\partial V}\right)_T = \frac{\partial}{\partial V} f(T, p(T,V)) = \frac{\partial f}{\partial p}\frac{\partial p}{\partial V}$$

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{\partial}{\partial T}(p(T,V)) = \frac{\partial p}{\partial T} + \frac{\partial p}{\partial V}\frac{\partial V}{\partial T} = 0$$

$$\frac{\partial V}{\partial T} = -\frac{\partial p}{\partial T} \Big/ \frac{\partial p}{\partial V}$$

$$\left(\frac{\partial E}{\partial T}\right)_V + \left(\frac{\partial E}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p =$$

$$= \frac{\partial f}{\partial T} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial T} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial V}\cdot\left(-\frac{\partial p}{\partial T}\Big/\frac{\partial p}{\partial V}\right) = \frac{\partial f}{\partial T}$$

2-41)

$$\begin{cases} u = xy \\ v = x/y \end{cases} \qquad x^2 = uv \qquad y^2 = u/v$$

$$z'_x = y = \sqrt{u/v}$$

$$(z'_x)'_u = \frac{1}{2v}(u/v)^{-1/2} = \frac{1}{2\sqrt{uv}}.$$

$$z'_u = 1, \quad (z'_u)'_x = 0.$$