6.1) $I = \iint_D \frac{dx\,dy}{3+x^2-y}$ $\quad D = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le 2\}$

a)

$$\frac{1}{7} = \frac{1}{3+2^2-0} \le \frac{1}{3+x^2-y} \le \frac{1}{3+0^2-2} = 1$$

$$\iint_D \frac{1}{7}\,dx\,dy = \frac{1}{7}\,\text{Area}(D) = \frac{1}{7}\cdot 2\cdot 2 \le \iint_D \frac{1}{3+x^2-y}\,dx\,dy$$

$$\le \iint_D 1\,dx\,dy = \text{Area}(D) = 2\cdot 2$$

D.v.s. $\frac{4}{7} \le I \le 4$.

b)

D_3	D_4
D_1	D_2

På D_1: $\frac{1}{4} \le \frac{1}{3+x^2-y} \le \frac{1}{2}$,

D_2: $\frac{1}{7} \le \frac{1}{3+x^2-y} \le \frac{1}{3}$

D_3: $\frac{1}{3} \le \frac{1}{3+x^2-y} \le 1$

D_4: $\frac{1}{6} \le \frac{1}{3+x^2-y} \le \frac{1}{2}$

Så, eftersom varje D_i har area 1 får vi

$$\iint_{D_1} \frac{1}{4}\,dx\,dy + \iint_{D_2} \frac{1}{7}\,dx\,dy + \iint_{D_3} \frac{1}{3}\,dx\,dy + \iint_{D_4} \frac{1}{6}\,dx\,dy \le$$

$$\le \iint_{D_1} \frac{1}{3+x^2-y}\,dx\,dy + \iint_{D_2} \frac{1}{3+x^2-y}\,dx\,dy + \iint_{D_3} \frac{1}{3+x^2-y}\,dx\,dy + \iint_{D_4} \frac{1}{3+x^2-y}\,dx\,dy =$$

$$= \iint_D \frac{1}{3+x^2-y}\,dx\,dy \le$$

$$\le \iint_{D_1} \frac{1}{2}\,dx\,dy + \iint_{D_2} \frac{1}{3}\,dx\,dy + \iint_{D_3} 1\,dx\,dy + \iint_{D_4} \frac{1}{2}\,dx\,dy,$$

$$\frac{1}{4} + \frac{1}{7} + \frac{1}{3} + \frac{1}{6} = \frac{25}{28} \le I \le \frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{2} = \frac{7}{3}$$

6.2)

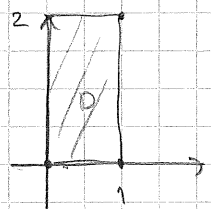
a) $\iint_D (x+y)^2\,dx\,dy =$ | $D=\{(x,y): |x|\le 1, |y|\le 1\} =$

$= \{(x,y): -1\le x\le 1, -1\le y\le 1\}$

$$= \int_{-1}^{1}\left(\int_{-1}^{1}(x+y)^2\,dx\right)dy = \int_{-1}^{1}\left[\frac{(x+y)^3}{3}\right]_{x=-1}^{1} dy =$$

$$= \int_{-1}^{1}\left(\frac{(1+y)^3}{3} - \frac{(-1+y)^3}{3}\right)dy = \left[\frac{(1+y)^4}{12} - \frac{(y-1)^4}{12}\right]_{y=-1}^{1} = \underline{\underline{\frac{8}{3}}}.$$

b)



$D=\{(x,y): 0\le x\le 1, 0\le y\le 2\}$

$$\iint_D \frac{dx\,dy}{1+x+y} = \int_0^2\left(\int_0^1 \frac{1}{1+x+y}\,dx\right)dy = \int_0^2 \left[\ln|1+x+y|\right]_{x=0}^{1} dy$$

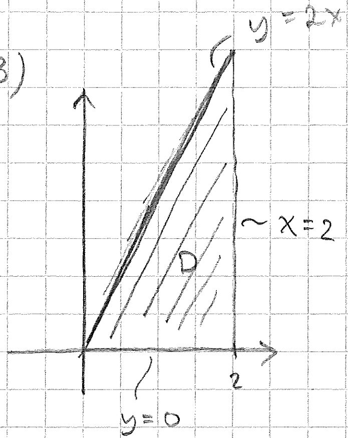
$$= \int_0^2 \left(\ln(2+y) - \ln(1+y)\right)dy \overset{*}{=} \left[(2+y)\ln(2+y) - (1+y)\ln(1+y)\right]_0^2 - \int_0^2 (1-1)\,dy =$$

$$= 4\ln 4 - 3\ln 3 - 2\ln 2 = \ln\frac{4^4}{3^3 2^2} = \ln\frac{64}{27}$$

$*$ Partiell integration

$$\int 1\cdot\ln(a+y)\,dy = /\,(a+y)'=1\,/ = (a+y)\ln(a+y) - \int\frac{a+y}{a+y}\,dy \ldots$$

6.3)



$D = \{(x,y) : 0 \leq x \leq 2,\ 0 \leq y \leq 2x\}$

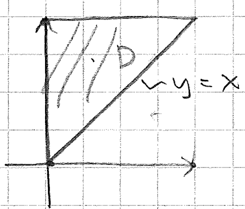
$= \{(x,y) : 0 \leq y \leq 4,\ \frac{y}{2} \leq x \leq 2\}$

$$\iint_D (xy + y^2)\, dx\, dy = \int_0^4 \left(\int_{y/2}^2 (xy+y^2)\, dx \right) dy =$$

$$= \int_0^2 \left(\int_0^{2x} (xy+y^2)\, dy \right) dx = \int_0^2 \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_{y=0}^{2x} dx =$$

$$= \int_0^2 \left(\frac{4x^3}{2} + \frac{8x^3}{3} \right) dx = \left[\frac{2x^4}{4} + \frac{2x^4}{3} \right]_0^2 = \ldots = \underline{\underline{\frac{56}{3}}}.$$

6.4a)



$D = \{(x,y) : 0 \leq y \leq 2,\ 0 \leq x \leq y\}$

$$\iint_D (x-y)e^{x+y}\, dx\, dy = \int_0^2 \left(\int_0^y (xe^x e^y - ye^x e^y)\, dx \right) dy =$$

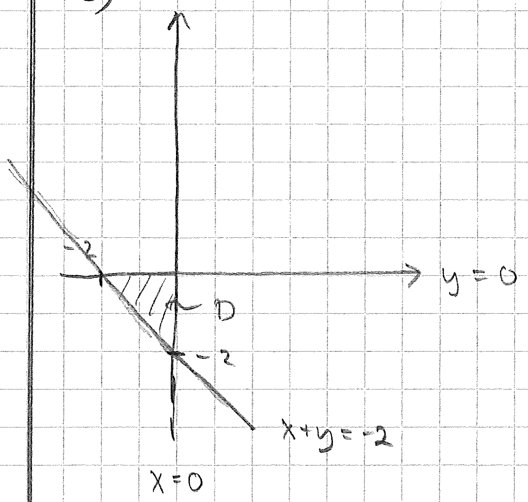
$$= \int_0^2 \left[xe^x e^y - e^x e^y - ye^x e^y \right]_{x=0}^{y} dx = \int_0^2 (ye^{2y} - e^{2y} - ye^{2y} + e^y + ye^y)\, dy$$

$$= \int_0^2 (ye^y + e^y - e^{2y})\, dy = \left[ye^y - e^y + e^y - \frac{e^{2y}}{2} \right]_0^2 = 2e^2 - \frac{e^4}{2} + \frac{1}{2}$$

$$= \underline{\underline{\frac{4e^2 - e^4 + 1}{2}}}$$

6.4)

b)



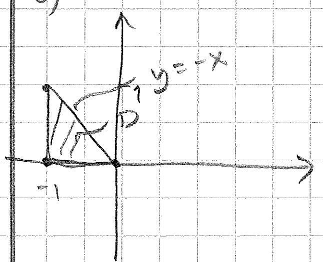
$D = \{(x,y) : -2 \le x \le 0,\ -2-x \le y \le 0\}$

$$\iint_D (2+x+y)\,dx\,dy = \int_{-2}^{0}\left(\int_{-2-x}^{0}(2+x+y)\,dy\right)dx = \int_{-2}^{0}\left[(2+x)y+\frac{y^2}{2}\right]_{y=-2-x}^{0}dx$$

$$= \int_{-2}^{0}\left(-(2+x)(-2-x)-\frac{(-2-x)^2}{2}\right)dx = \int_{-2}^{0}\frac{(2+x)^2}{2}\,dx = \left[\frac{(2+x)^3}{6}\right]_{-2}^{0}$$

$$= \frac{8}{6} = \underline{\underline{\frac{4}{3}}}$$

c)



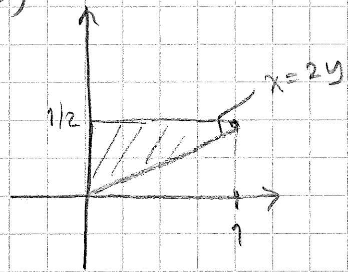
$D = \{(x,y) : -1 \le x \le 0,\ 0 \le y \le -x\}$

$$\iint_D e^{x^2}\,dx\,dy = \int_{-1}^{0}\left(\int_{0}^{-x} e^{x^2}\,dy\right)dx = \int_{-1}^{0} -xe^{x^2}\,dx = \left[-\frac{e^{x^2}}{2}\right]_{-1}^{0}$$

$$= -\frac{1}{2}+\frac{e}{2} = \underline{\underline{\frac{e-1}{2}}}.$$

6.4)

d)



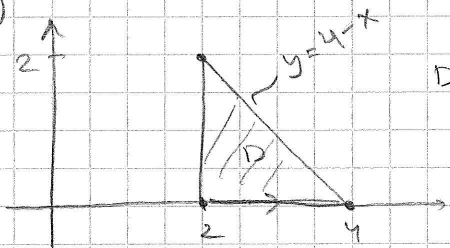
$D = \{(x,y) : 0 \le y \le 1/2,\ 0 \le x \le 2y\}$

$$\iint_D \frac{x^3}{1+y^5}\,dx\,dy = \int_0^{1/2}\left(\int_0^{2y} \frac{x^3}{1+y^5}\,dx\right)dy =$$

$$= \int_0^{1/2} \left[\frac{x^4}{4(1+y^5)}\right]_{x=0}^{2y} dy = \int_0^{1/2} \frac{4y^4}{1+y^5}\,dy =$$

$$= \left[\frac{4}{5}\ln(1+y^5)\right]_0^{1/2} = \frac{4}{5}\ln\left(1+\frac{1}{32}\right) = \underline{\underline{\frac{4}{5}\ln\frac{33}{32}}}$$

e)



$D = \{(x,y) : 2 \le x \le 4,\ 0 \le y \le 4-x\}$

$$\iint_D \frac{dx\,dy}{x^2+2x} = \int_2^4\left(\int_0^{4-x} \frac{1}{x^2+2x}\,dy\right)dx = \int_2^4 \frac{4-x}{x^2+2x}\,dx$$

$$= \int_2^4 \left(\frac{4}{x(x+2)} - \frac{1}{x+2}\right)dx = \int_2^4\left(\frac{2}{x} - \frac{2}{x+2} - \frac{1}{x+2}\right)dx =$$

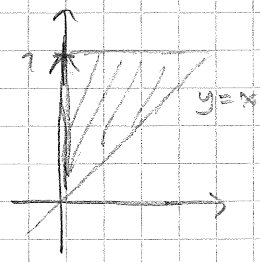
$$= \Big[2\ln x - 3\ln(x+2)\Big]_2^4 = 2\ln 4 - 3\ln 6 - 2\ln 2 + 3\ln 4 =$$

$$= 4\ln 4 - 3\ln 6 = \ln\frac{4^4}{6^3} = \ln\frac{4\cdot 2^3}{3^3} = \underline{\underline{\ln\frac{32}{27}}}$$

6.4)

f)

$$\iint_D \frac{x+2xy}{1+y^2}\,dx\,dy =$$



$D = \{(x,y): 0 \le y \le 1,\ 0 \le x \le y\}$

$$= \int_0^1 \left(\int_0^y \frac{x+2xy}{1+y^2}\,dx \right) dy =$$

$$= \int_0^1 \left[\frac{x^2\left(\frac{1}{2}+y\right)}{1+y^2} \right]_{x=0}^{y} dy = \int_0^1 \frac{y^2\left(\frac{1}{2}+y\right)}{1+y^2}\,dy =$$

$$= \frac{1}{2}\int_0^1 \frac{y^2+2y^3}{1+y^2}\,dy = \frac{1}{2}\int_0^1 \frac{2y(y^2+1)-2y+y^2}{1+y^2}\,dy =$$

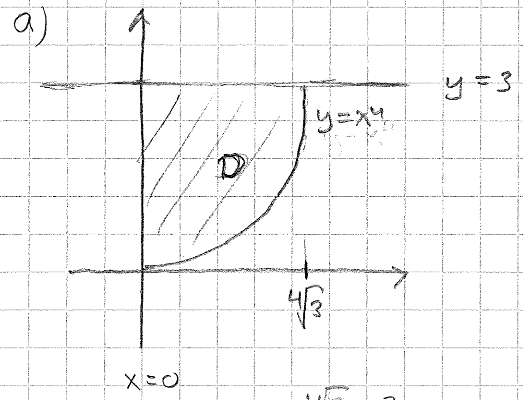
$$= \frac{1}{2}\int_0^1 \left(2y - \frac{2y}{1+y^2} + 1 - \frac{1}{1+y^2}\right) dy =$$

$$= \frac{1}{2}\left[y^2 - \ln(1+y^2) + y - \arctan y \right]_0^1 =$$

$$= \frac{1}{2}\left(2 - \ln 2 - \frac{\pi}{4}\right) = 1 - \frac{\ln 2}{2} - \frac{\pi}{8}.$$

6.5)

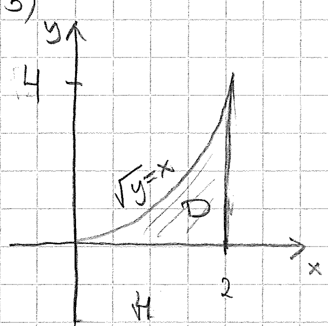
a)



$$\iint_D x^3 y \, dx\,dy = \int_0^{\sqrt[4]{3}} \left(\int_{x^4}^{3} x^3 y \, dy \right) dx = \int_0^{\sqrt[4]{3}} \left[\frac{x^3 y^2}{2} \right]_{y=x^4}^{3} dx =$$

$$= \int_0^{\sqrt[4]{3}} \left(\frac{9x^3}{2} - \frac{x^{11}}{2} \right) dx = \left[\frac{9x^4}{8} - \frac{x^{12}}{24} \right]_0^{\sqrt[4]{3}} = \frac{27}{8} - \frac{27}{24} = \frac{54}{24} = \underline{\underline{\frac{9}{4}}}$$

b)



$$\iint_D x \cos\sqrt{y} \, dx\,dy = \int_0^4 \left(\int_{\sqrt{y}}^{2} x \cos\sqrt{y} \, dx \right) dy =$$

$$= \int_0^4 \left[\frac{x^2}{2} \cos\sqrt{y} \right]_{x=\sqrt{y}}^{2} dy = \int_0^4 \left(2\cos\sqrt{y} - \frac{y}{2}\cos\sqrt{y} \right) dy =$$

$$= \Big/ \begin{matrix} \sqrt{y} = t, \; y = t^2 \\ dy = 2t\,dt \\ 0 \mapsto 0, \; 4 \mapsto 2 \end{matrix} \Big/ = \int_0^2 \left(2\cos t - \frac{t^2}{2}\cos t \right) 2t \, dt =$$

$$= \int_0^2 (4t - t^3) \cos t \, dt = \Big[(4t - t^3) \sin t \Big]_0^2 - \int_0^2 (4 - 3t^2) \sin t \, dt$$

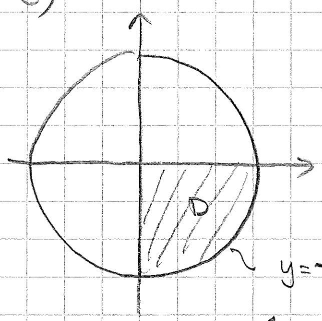
$$= \Big[(4t - t^3)\sin t + (4 - 3t^2)\cos t \Big]_0^2 + \int_0^2 6t \cos t \, dt =$$

$$= \Big[(4t - t^3)\sin t + (4 - 3t^2)\cos t + 6t \sin t + 6\cos t \Big]_0^2 = \ldots =$$

$$= 12 \sin 2 - 2\cos 2 - 10$$

6.5)

c)

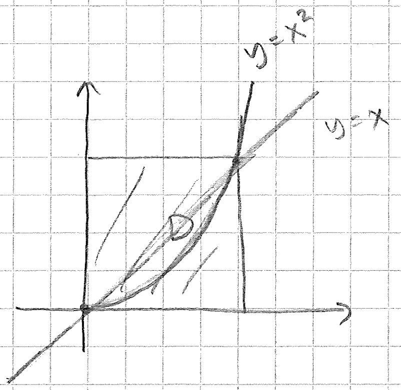


$y = -\sqrt{1-x^2}$ på "undre" delen

$$\iint_D xy\,dx\,dy = \int_0^1 \left(\int_{-\sqrt{1-x^2}}^{0} xy\,dy \right) dx = \int_0^1 \left[\frac{xy^2}{2} \right]_{y=-\sqrt{1-x^2}}^{0} dx =$$

$$= \int_0^1 \frac{-x(1-x^2)}{2}\,dx = \int_0^1 \frac{-x+x^3}{2}\,dx = \left[-\frac{x^2}{4} + \frac{x^4}{8} \right]_0^1 = \underline{\underline{-\frac{1}{8}}}\,.$$

6.6)



a) $|x-y| = (x-y)$ då $x \geq y$, $|x-y| = y-x$ då $y \geq x$.

$$\iint_D |x-y|\,dx\,dy = \int_0^1 \left(\int_y^1 (x-y)\,dx \right) dy + \int_0^1 \left(\int_0^y y-x \; dx \right) dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - xy \right]_{x=y}^1 dy + \int_0^1 \left[xy - \frac{x^2}{2} \right]_{x=0}^y dy =$$

$$= \int_0^1 \left(\frac{1}{2} - y + \frac{y^2}{2} \right) dy + \int_0^1 \left(\frac{y^2}{2} \right) dy =$$

$$= \left[\frac{y}{2} - \frac{y^2}{2} + \frac{y^3}{6} \right]_0^1 + \left[\frac{y^3}{6} \right]_0^1 = \frac{1}{6} + \frac{1}{6} = \underline{\underline{\frac{1}{3}}}$$

b)

$$\iint_D \max(x^2, y)\,dx\,dy = \int_0^1 \left(\int_0^1 \max(x^2,y)\,dy \right) dx =$$

$$= \int_0^1 \left(\int_0^{x^2} \max(x^2,y)\,dy + \int_{x^2}^1 \max(x^2,y)\,dy \right) dx =$$

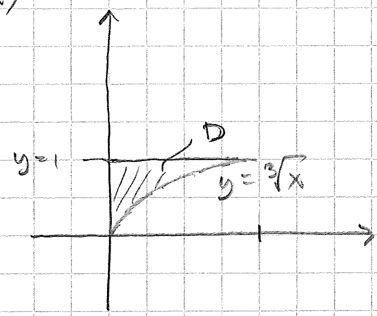
$$= \int_0^1 \left(\int_0^{x^2} x^2\,dy \right) dx + \int_0^1 \left(\int_{x^2}^1 y\,dy \right) dx = \int_0^1 x^4\,dx + \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^2}^1 dx$$

$$= \frac{1}{5} + \int_0^1 \left(\frac{1}{2} - \frac{x^4}{2} \right) dx = \frac{1}{5} + \frac{1}{2} - \frac{1}{10} = \underline{\underline{\frac{3}{5}}}$$

6.8)

a)

$$D=\{(x,y):\ 0\le x\le 1,\ \sqrt[3]{x}\le y\le 1\}=$$
$$=\{(x,y):\ 0\le y\le 1,\ 0\le x\le y^3\}$$

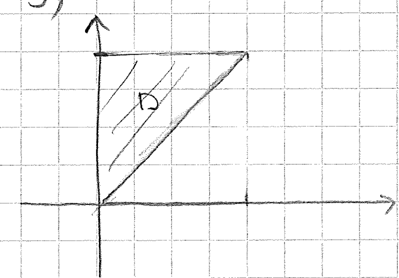


$$\int_0^1\left(\int_{\sqrt[3]{x}}^{1}\frac{dy}{\sqrt{1+y^8}}\right)dx=\int_0^1\left(\int_0^{y^3}\frac{dx}{\sqrt{1+y^8}}\right)dy=$$

$$=\int_0^1\frac{y^3}{\sqrt{1+y^8}}\,dy=\left/\begin{array}{c}y^4=t\\4y^3dy=dt\\0\mapsto 0,\ 1\mapsto 1\end{array}\right/=\int_0^1\frac{1}{\sqrt{1+t^2}}\,\frac{dt}{4}=$$

$$=\frac{1}{4}\left[\ln\left|t+\sqrt{1+t^2}\right|\right]_0^1=\underline{\underline{\frac{1}{4}\ln(1+\sqrt{2})}}$$

b)



$$\int_0^1\left(\int_0^y\frac{y\,dx}{(4-x^2-y^2)^{3/2}}\right)dy=\int_0^1\left(\int_x^1\frac{y}{(4-x^2-y^2)^{3/2}}\,dy\right)dx=$$

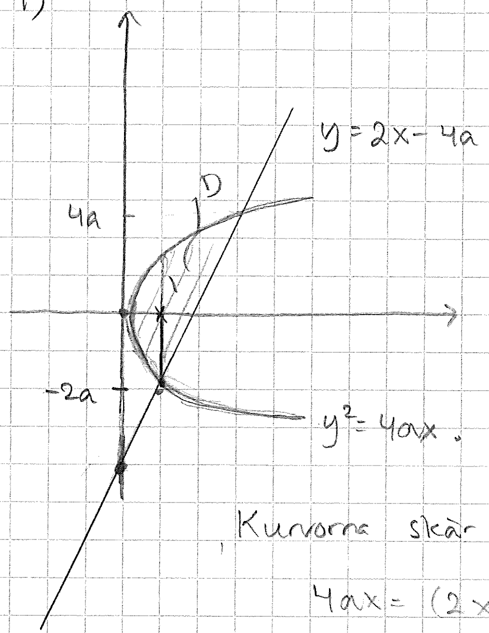
$$=\int_0^1\left[(4-x^2-y^2)^{-1/2}\right]_{y=x}^{1}dx=\int_0^1\left((3-x^2)^{-1/2}-(4-2x^2)^{-1/2}\right)dx$$

$$=-\frac{1}{2}\int_0^1\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}\,dx+\frac{1}{\sqrt{3}}\int_0^1\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}\,dx=$$

$$=-\frac{1}{2}\left[\sqrt{2}\arcsin\left(\frac{x}{\sqrt{2}}\right)\right]_0^1+\frac{1}{\sqrt{3}}\left[\sqrt{3}\arcsin\left(\frac{x}{\sqrt{3}}\right)\right]_0^1=$$

$$=-\frac{1}{\sqrt{2}}\arcsin\left(\frac{1}{\sqrt{2}}\right)+\arcsin(1/\sqrt{3})=\underline{\underline{-\frac{\pi}{4\sqrt{2}}+\arcsin(1/\sqrt{3})}}$$

6.7)



Kurvorna skär varandra då

$$4ax = (2x-4a)^2 = 4x^2 - 16ax + 16a^2$$

$$\Leftrightarrow \quad x^2 - 5ax + 4a^2 = 0$$

$$\Leftrightarrow \quad x = \frac{5a}{2} \pm \sqrt{\frac{25a^2 - 16a^2}{4}} \quad , \; x = a, \; x = 4a$$

Vi kan antingen dela upp området i två delar om vi vill ha fixa gränser för x, eller ha fixa gränser för y: $x = a$ motsvarar $y = -2a$, $x = 4a$ motsvarar $y = 4a$.

$$D = \left\{ (x,y) : -2a \le y \le 4a, \; \frac{y^2}{4a} \le x \le \frac{y+4a}{2} \right\}$$

$$\int_{-2a}^{4a} \left(\int_{\frac{y^2}{4a}}^{\frac{y+4a}{2}} xy \, dx \right) dy = \int_{-2a}^{4a} \left[\frac{x^2 y}{2} \right]_{x=\frac{y^2}{4a}}^{\frac{y+4a}{2}} dy =$$

$$= \int_{-2a}^{4a} \left(\frac{\left(\frac{y+4a}{2}\right)^2 y}{2} - \frac{\left(\frac{y^2}{4a}\right)^2 y}{2} \right) dy = \frac{1}{8} \int_{-2a}^{4a} \left(y^3 + 8ay^2 + 16a^2 y - \frac{y^5}{4a^2} \right) dy$$

$$= \frac{1}{8} \left[\frac{y^4}{4} + 8a\frac{y^3}{3} + 8a^2 y^2 - \frac{y^6}{24a^2} \right]_{y=-2a}^{4a} =$$

$$= \ldots = \underline{\underline{\frac{45a^4}{2}}}$$