1.21 a)

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \Big/ \begin{matrix} x=\rho\cos\varphi \\ y=\rho\sin\varphi \end{matrix} \Big/ = \lim_{\rho\to 0} \frac{\rho^3(\cos^3\varphi+\sin^3\varphi)}{\rho^2}$$

$$= \lim_{\rho\to 0} \rho\underbrace{(\cos^3\varphi+\sin^3\varphi)}_{\text{Begränsad}} = \underline{\underline{0}}.$$

SVAR: 0

b)

$$\frac{x^2-2y^2}{2x^2+y^2} = f(x,y)$$

Om vi går längs kurvan $x=0$ mot origo får vi

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{-2y^2}{y^2} = -2.$$

Om vi går längs kurvan $y=0$ mot origo får vi

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

Alltså existerar inte gränsvärdet

SVAR: Existerar ej

c)

$$\lim_{(x,y)\to(0,0)} \frac{x^3-2y^3}{2x^2+y^2} = \lim_{\rho\to 0} \frac{\rho^3(\cos^3\varphi-2\sin^3\varphi)}{\rho^2(2\cos^2\varphi+\sin^2\varphi)} =$$

$$= \lim_{\rho\to 0} \rho\left(\frac{\cos^3\varphi-2\sin^3\varphi}{\cos^2\varphi+1}\right) = 0$$

eftersom $1+\cos^2\varphi \geq 1$ så ä $\dfrac{\cos^3\varphi-2\sin^3\varphi}{\cos^2\varphi+1}$ begränsad.

SVAR: 0

d)

$$f(x,y) = \frac{x^2}{y-x^2}$$

Längs $y=0$: $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2}{-x^2} = -1.$

Längs $y=2x^2$: $\lim_{x\to 0} f(x,2x^2) = \lim_{x\to 0} \frac{x^2}{2x^2-x^2} = 1.$

(OBS! $y=2x^2$, $x\neq 0$ ligger i D_f).

SVAR: Existerar ej.

1.21 e)

$$\lim_{(x,y)\to(0,0)} \frac{2x^3 - xy^2}{x^2+y^2-xy} = \lim_{\rho\to 0} \frac{\rho^3(2\cos^3\varphi - \cos\varphi\sin^2\varphi)}{\rho^2(\cos^2\varphi+\sin^2\varphi - \cos\varphi\sin\varphi)} =$$

$$= \lim_{\rho\to 0} \rho \left(\frac{2\cos^3\varphi \quad \cos\varphi\sin^2\varphi}{1-\cos\varphi\sin\varphi}\right) =$$

$$= \Big/ \; 1-\cos\varphi\sin\varphi = 1 - \frac{\sin 2\varphi}{2} \geq 1/2 \Big/ = 0 .$$

SVAR: 0.

f)

$$f(x,y) = \frac{2x^3 - xy^2}{x^2+y^2-2xy} = \frac{x(2x^2-y^2)}{(x-y)^2}$$

Ej def längs $y = x$.

Längs kurvan $y = \sqrt{2x^2}$ får vi

$$\lim_{x\to 0} f(x, \sqrt{2x^2}) = \lim_{x\to 0} \frac{x(2x^2 - 2x^2)}{(x-\sqrt{2x^2})} = 0.$$

Längs kurvan $y = x + x^2$, $x > 0$ t.ex.

(som ligger i D$_f$) får vi

$$\lim_{x\to 0^+} f(x, x+x^2) = \lim_{x\to 0^+} \frac{x(2x^2 - (x+x^2)^2)}{(x-x-x^2)^2} =$$

$$= \lim_{x\to 0^+} \frac{x(x^2 - 2x^3 - x^4)}{x^4} = \infty .$$

SVAR: EXISTERAR EJ.

1.22 a)

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{\rho\to 0} \frac{\sin \rho^2}{\rho^2} = 1,$$

eftersom $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

SVAR: 1

b) $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2+x^2y} = \Big/ \sin t = t + O(t^3) \Big/ =$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2 + O((x^2+y^2)^3)}{x^2+y^2+x^2y} =$$

$$= \lim_{\rho\to 0} \frac{\rho^2 + O(\rho^6)}{\rho^2 + \rho^3\cos^2\varphi\sin\varphi} =$$

$$= \lim_{\rho\to 0} \frac{1 + O(\rho^4)}{1 + \underbrace{\rho\cos^2\varphi\sin\varphi}_{\to 0}} = 1.$$

SVAR: 1.

c) $$\lim_{(x,y)\to(0,0)} \frac{x+y+1}{\ln(x^2+2y^2)} = 0$$

eftersom $x+y+1 \to 1$ och $\ln(x^2+2y^2) \to -\infty$.

SVAR: 0

1.23 a)

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{x^2+2y^2-2x+1} = \Big/ x = 1+t \Big/ =$$

$$= \lim_{(t,y)\to(0,0)} \frac{(1+t)y-y}{(1+t)^2+2y^2-2(1+t)+1} =$$

$$= \lim_{(t,y)\to(0,0)} \frac{ty}{t^2+2t+1+2y^2-2-2t+1} =$$

$$= \lim_{(t,y)\to(0,0)} \frac{ty}{t^2+2y^2} = \Big/ \begin{matrix} t=\rho\cos\varphi \\ y=\rho\sin\varphi \end{matrix} \Big/ =$$

$$= \lim_{\rho\to 0} \frac{\rho^2\cos\varphi\sin\varphi}{\rho^2(\cos^2\varphi+2\sin^2\varphi)} = \lim_{\rho\to 0} \frac{\cos\varphi\sin\varphi}{1+\sin^2\varphi}.$$

Beror på φ: $\varphi = 0$ ger $\frac{\cos 0\sin 0}{1+\sin^2 0} = 0$,

$\varphi = \pi/4$ ger $\frac{\cos\pi/4 \cdot \sin\pi/4}{1+\sin^2\pi/4} = \frac{1/2}{1+1/2} = \frac{1}{3}$

SVAR: Existerar ej.

(Alt: Längs $y=0$: $\lim_{x\to 1} f(x,0) = \lim_{x\to 0} \frac{0}{x^2-2x+1} = 0$

Längs $y=x-1$: $\lim_{x\to 1} f(x,x-1) = \frac{x(x-1)-(x-1)}{x^2+2(x-1)^2-2x+1} =$

$= \lim_{x\to 1} \frac{(x-1)^2}{3(x-1)^2} = \frac{1}{3}$.)

b) $$\lim_{(x,y)\to(1,0)} \frac{xy^2-y^2}{x^2+2y^2-2x+1} = \lim_{(t,y)\to(0,0)} \frac{(1+t)y^2-y^2}{(1+t)^2+2y^2-2(1+t)+1} =$$

$$= \lim_{(t,y)\to(0,0)} \frac{ty^2}{t^2+2y^2} = \Big/ \begin{matrix} t=\rho\cos\varphi \\ y=\rho\sin\varphi \end{matrix} \Big/ =$$

$$= \lim_{\rho\to 0} \frac{\rho^3\cos\varphi\sin^2\varphi}{\rho^2(\cos^2\varphi+2\sin^2\varphi)} = \lim_{\rho\to 0} \rho \underbrace{\left(\frac{\cos\varphi\sin^2\varphi}{1+\sin^2\varphi}\right)}_{\text{beg}} = 0$$

SVAR: 0.

1.24 a)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} \quad \left/ \begin{array}{l} x = r\cos\varphi\sin\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta \end{array} \right/$$

$$= \lim_{r\to 0} \frac{r^3 \cos\varphi\sin\varphi\sin^2\theta\cos\theta}{r^2} =$$

$$= \lim_{r\to 0} r \underbrace{(\cos\varphi\sin\varphi\sin^2\theta\cos\theta)}_{\text{beg.}} = 0$$

SVAR: 0

b)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{3xz^2}{x^2+2y^2+3z^2} =$$

$$= \lim_{r\to 0} \frac{3r^3\cos\varphi\sin\theta\cos^2\theta}{r^2(\cos^2\varphi\sin^2\theta + 2\sin^2\varphi\sin^2\theta + 3\cos^2\theta)}$$

$$= \lim_{r\to 0} 3r \frac{\cos\varphi\sin\theta\cos^2\theta}{1+\sin^2\varphi\sin^2\theta + 2\cos^2\theta} = 0$$

(eftersom $1+\sin^2\varphi\sin^2\theta + 2\cos^2\theta \geq 1$.)

c)

$$f(x,y,z) = \frac{x+2y-z}{3x^2+y^2+z^2}.$$

Längs kurva $y=z=0$, $x>0$ får vi

$$\lim_{x\to 0^+} f(x,0,0) = \lim_{x\to 0^+} \frac{x}{3x^2} = +\infty$$

Längs kurva $y=z=0$, $x<0$ får vi

$$\lim_{x\to 0^-} f(x,0,0) = \lim_{x\to 0^-} \frac{x}{3x^2} = -\infty$$

SVAR: Existerar ej.

1.25 a)

$$\lim_{\sqrt{x^2+y^2}\to\infty} \frac{\sin x^2y^2}{2x^2+3y^2} = 0$$

eftersom $|\sin x^2y^2| \le 1$ och $2x^2+3y^2 \to \infty$.

SVAR: 0

b)

$$\lim_{\sqrt{x^2+y^2}\to\infty} \frac{x^2+y^2}{x^2+y^2+x} = \lim_{\sqrt{x^2+y^2}\to\infty} \frac{1}{1+\frac{x}{x^2+y^2}} = \frac{1}{1} = 1.$$

(Vi använde att $\frac{x}{x^2+y^2} = \frac{\cos\varphi}{\rho} \to 0$ då $\rho\to\infty$)

SVAR: 1.

c)

$$\lim_{\sqrt{x^2+y^2}\to\infty} xye^{-x^2-y^2} = \lim_{\rho\to\infty} \frac{\rho^2\cos\varphi\sin\varphi}{e^{\rho^2}} = 0$$

eftersom $\frac{t}{e^t} \to 0$ då $t\to\infty$,

och $\cos\varphi\sin\varphi$ ä begränsad.

SVAR: 0

1.28)

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} x^2+y^2 = 0 \neq f(0,0) = 1.$$

Så inte kontinuerlig i (0,0)

Kontinuerlig i alla andra punkter

(för att visa det:

$$\lim_{(h,k)\to(0,0)} f(a+h, b+k) = \lim_{(h,k)\to(0,0)} \left((a+h)^2+(b+k)^2\right) =$$

$$= \lim_{(h,k)\to(0,0)} \left(a^2+2ah+h^2+b^2+2bk+k^2\right) = a^2+b^2 = f(a,b)$$

om $(a,b) \neq (0,0)$.)

Om vi definierar $f(0,0)=0$ istället så blir

f kontinuerlig överallt.

1.29) Frågan är helt enkelt om

$\lim_{(x,y)\to(0,0)} f(x,y)$ existerar.

Om det gör det blir f kontinuerlig om vi definierar

$$f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y).$$

a) $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \left/ t = x^2+y^2 \right/ = \lim_{t\to 0} \frac{\sin t}{t} = 1.$$

SVAR: Ja, $f(0,0)=1$.

b) $$f(x,y) = \frac{(2x+y)^2}{x^2+3y^2+2xy}$$

Längs $x=0$: $$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{y^2}{3y^2} = \frac{1}{3}$$

Längs $y=-2x$: $$\lim_{x\to 0} f(x,-2x) = \lim_{x\to 0} \frac{0^2}{x^2+3(-2x)^2+2x(-2x)}$$

$$= \lim_{x\to 0} \frac{0^2}{9x^2} = 0.$$

SVAR: Nej.

c) $$\lim_{(x,y)\to(0,0)} \frac{6x^2+3y^2+x^2y^2}{2x^2+y^2} =$$

$$\lim_{(x,y)\to(0,0)} \left(\frac{3(2x^2+y^2)}{2x^2+y^2} + \frac{x^2y^2}{2x^2+y^2} \right) =$$

$$= \lim_{(x,y)\to(0,0)} \left(3 + \frac{x^2y^2}{2x^2+y^2} \right) =$$

$$= \lim_{\rho\to 0} \left(3 + \frac{\rho^4 \cos^2\varphi \sin^2\varphi}{\rho^2(2\cos^2\varphi + \sin^2\varphi)} \right) =$$

$$= \lim_{\rho\to 0} \left(3 + \rho^2 \underbrace{\left(\frac{\cos^2\varphi \sin^2\varphi}{1+\cos^2\varphi} \right)}_{\text{beg}} \right) = 3.$$

SVAR: JA, $f(0,0)=3$.

d) $$\lim_{(x,y)\to(0,0)} x\exp\left(\frac{-1}{\sqrt{x^2+y^2}} \right) = \lim_{\rho\to 0} \frac{\rho\cos\varphi}{e^{1/\rho}} = 0$$

SVAR: JA, $f(0,0)=0$.

1.20 a)

$$|x^2+(y-2)| \le |x^2| + |y-2| \le |x^2+(y-2)^2| + \sqrt{(y-2)^2+x^2}$$

$$= \rho^2 + \rho = g(\rho).$$

b)

$$|\sin(x-y)| \le |x-y| \le |x|+|y| \le \sqrt{x^2+y^2} + \sqrt{x^2+y^2} = 2\rho = g(\rho)$$

c)

$$\left| x + \frac{2}{y} - 2 \right| \le |x-1| + \left|\frac{2}{y} - 1\right| = |x-1| + \left|\frac{2-y}{y}\right|$$

$$\le / \text{Om } y \ge 1 / \le |x-1| + |y-2|$$

$$\le \sqrt{(x-1)^2+(y-2)^2} + \sqrt{(x-1)^2+(y-2)^2} = 2\rho = g(\rho).$$

1.26 a)

$y = kx$ ger, då $f(x,y) = \frac{y^4}{y^4+(y-x^2)^2}$, $k \ne 0$,

$$\lim_{x\to 0} f(x,kx) = \frac{k^4x^4}{k^4x^4+(kx-x^2)^2} =$$

$$= \lim_{x\to 0} \frac{k^4x^4}{k^2x^2+(k^2+1)x^4-2kx^3} = 0.$$

Längs $y=0$:

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{(-x^2)^2} = \lim_{x\to 0} \frac{0}{x^4} = 0.$$

Längs $x=0$:

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{y^4}{y^4+y^2} = 0.$$

b)

Längs $y=x^2$

$$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{x^8}{x^8} = 1 \ne 0.$$

1.27 a)

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{1+x^2}-\sqrt{1-y^2}}{x^2+y^2} = \Big/ \sqrt{1+t} = 1 + t/2 + O(t^2) \Big/$$

$$= \lim_{(x,y)\to(0,0)} \frac{\left(1 + x^2/2 + O(x^4)\right) - \left(1 - y^2/2 + O(y^4)\right)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{1}{2}(x^2+y^2) + \left(O(x^4)+O(y^4)\right)}{x^2+y^2} = \underline{\underline{\frac{1}{2}}}.$$

b) $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{|x|+|y|} = \lim_{\rho\to 0} \frac{\rho^2}{\rho(|\cos\varphi|+|\sin\varphi|)}$$

$$= \Big/ |\cos\varphi| + |\sin\varphi| \geq 1 \Big/ = \underline{\underline{0}}.$$

c) $$f(x,y) = \frac{xy^2}{x^2+y^4}.$$

Längs $x=y^2$: $\lim_{y\to 0} f(y^2,y) = \frac{y^4}{y^4+y^4} = \frac{1}{2}$.

Längs $x=0$: $\lim_{y\to 0} f(0,y) = \cancel{\frac{y^4}{y^4} = 1}.$

$= \lim_{y\to 0} \frac{0}{y^4} = 0.$ Existerar ej.

1.30)

a) $$f(x,y,z) = \frac{x^2+y^3+z^4}{2x^2+y^2+3z^2}, \quad (x,y,z) \neq (0,0,0)$$

Längs $x=y=0$.

$$\lim_{z\to 0} f(0,0,z) = \lim_{z\to 0} \frac{z^4}{3z^2} = 0.$$

Längs $y=z=0$.

$$\lim_{x\to 0} f(x,0,0) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

SVAR: NEJ.

b) $$f(x,y,z) = \frac{xyz+yz}{x^2+y^2+z^2+2x+1} \quad (x,y,z) \neq (-1,0,0)$$

$$= \frac{(x+1)yz}{(x+1)^2+y^2+z^2}$$

$$\lim_{(x,y,z)\to(-1,0,0)} f(x,y,z) = \lim_{(t,y,z)\to(0,0,0)} f(-1+t,y,z) =$$

$$= \lim_{(t,y,z)} \frac{tyz}{t^2+y^2+z^2} = \lim_{r\to 0} \frac{r^3\cos\varphi\sin\varphi\sin^2\theta\cos\theta}{r^2}$$

$$= \lim_{r\to 0} r \underbrace{\cos\varphi\sin\varphi\sin^2\theta\cos\theta}_{\text{beg.}} = 0.$$

SVAR: JA, $f(-1,0,0)=0$.