Exempel

Bestäm alla funktioner av klass $\mathcal{C}^1$ i första kvadranten som uppfyller

$$2x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = y^2,$$

och $z(1, y) = y^2$, genom att införa

$$\begin{cases} u = xy^2, \\ v = x. \end{cases}$$

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Allmän lösning: $z = -y^2/2 + h(xy^2)$ där $h :]0, \infty[\to \mathbb{R}$ är av klass $\mathcal{C}^1$.

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SVAR: $z(x,y) = -y^2/2 + 3xy^2/2$.