Exempel

Bestäm alla lösningar z av klass $\mathcal{C}^2$ till

$$xz''_{xx} - yz''_{xy} + z'_x = 0, \quad y \neq 0,$$

genom att göra variabelbytet

$$\begin{cases} u = y, \\ v = xy. \end{cases}$$

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&= y\left(y\frac{\partial^2 z}{\partial v^2}\right) = y^2\frac{\partial^2 z}{\partial v^2}.
\end{aligned}
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$$\frac{\partial^2 z}{\partial x \partial y}$$

z''_{xy}

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&= \Big/ w = \frac{\partial z}{\partial v} \text{ i } (**) \Big/ \\
&= y\left(\frac{\partial^2 z}{\partial u \partial v} + x\frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial v} \\
&= xy\frac{\partial^2 z}{\partial v^2} + y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v}.
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SVAR: Alla funktioner på formen $z = H(y) + K(xy)$ där H, K är av klass $\mathcal{C}^2$ på $\mathbb{R}$.