Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinary Differential Equations and Dynamical Systems

Take-home examination 2021-01-14, 8.00–13.00

Rules, short version.

- Aids are permitted, but no collaboration with other persons is allowed.

Rules, long version.

- This is an individual examination, so you are required to answer the questions on your own.
- You may ask the teacher for clarifications (email **hans.lundmark@liu.se**). Except for that, it is not allowed to communicate in any way with other persons regarding the solutions of the problems during the exam. So you may not get help from others, and it is also not allowed to *give* help to other students who are taking this exam, for example by letting them look at your solutions.
- You can use any aids (books, computers, etc.), but you are expected to present your solutions with as much detail as if calculating by hand (like on a usual exam without aids). Consulting old information from online forums is allowed, but you may not make use of any questions or answers posted during the exam. Cite your sources whenever appropriate, and avoid quoting text verbatim; it is much preferred if you use your own formulations.
- The solutions should be handwritten (unless you have a special permit from LiU's disability coordinator to write on a computer). Writing by hand on a tablet is fine, but please use dark text on a white background.
- You may write your answers in English or in Swedish (or some mixture thereof).

You will find the problems **on the next page**.

Each problem will be marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Solve the initial value problem

$$\dot{x} = x - 2y, \qquad x(0) = 0,$$
$$\dot{y} = x - y, \qquad y(0) = 1,$$

and draw the solution curve $(x(t), y(t))$ in the xy-plane.

(Try to draw as accurately as possible. In particular, the nullclines of the system should be taken into account.)

2. Use linearization to classify the equilibrium points of the system

$$\dot{x} = y - x^2 - x, \qquad \dot{y} = y - x - 1,$$

and sketch the phase portrait.

3. Determine $k \in \mathbf{R}$ such that $x(t) = e^{kt}$ is a solution of the ODE

$$(t^2 + 1)\,\ddot{x}(t) - 2(t^2 + t + 1)\,\dot{x}(t) + 4t\,x(t) = 0,$$

and then use reduction of order to find the general solution.

4. (a) Determine a constant of motion $F(x, y)$ for the system

$$\dot{x} = xy, \qquad \dot{y} = 1 - x^2.$$

   (b) Determine all **linear** systems in $\mathbf{R}^2$ for which $F(x, y) = x^2 + y^2$ is a constant of motion.

   (c) Give an example of a **nonlinear** system in $\mathbf{R}^2$ for which $F(x, y) = x^2 + y^2$ is a constant of motion.

5. Show that the origin is a stable equilibrium for the system

$$\dot{x} = -y + xy, \qquad \dot{y} = x - x^2 - y^3.$$

(Hint: $V(x, y) = x^2 + y^2$.) Is it *asymptotically* stable? If so, is it *globally* asymptotically stable?

6. Construct polynomials $X(x, y)$ and $Y(x, y)$ such that the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ are stable limit cycles for the system $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$.

Solutions for TATA71 2021-01-14

1. The system matrix $\left(\begin{smallmatrix}1 & -2\\ 1 & -1\end{smallmatrix}\right)$ has the eigenvalues $\lambda = \pm i$, with eigenvectors $\left(\begin{smallmatrix}1\pm i\\ 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\ 1\end{smallmatrix}\right) \pm i\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$, so the change of variables

$$\begin{pmatrix}x\\ y\end{pmatrix} = u\begin{pmatrix}1\\ 0\end{pmatrix} + v\begin{pmatrix}1\\ 1\end{pmatrix}$$

brings the system to a canonical form which we know how to solve:

$$\begin{pmatrix}\dot u\\ \dot v\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{pmatrix}u(t)\\ v(t)\end{pmatrix} = \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}.$$

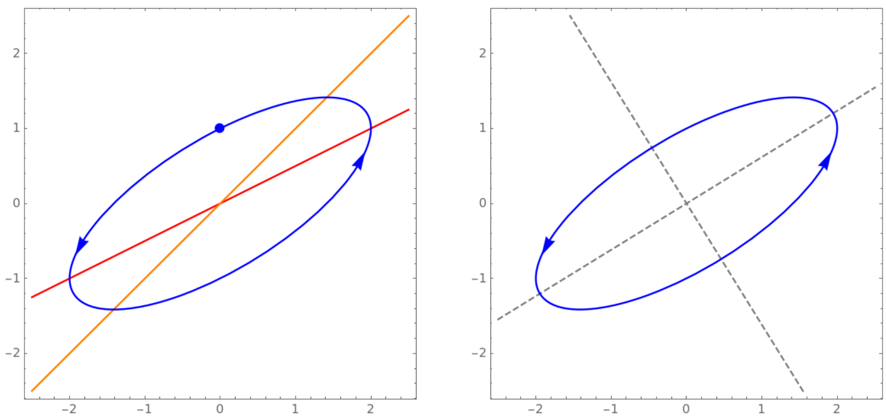
Thus, the general real-valued solution in the original variables is

$$\begin{aligned}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} &= u(t)\begin{pmatrix}1\\ 0\end{pmatrix} + v(t)\begin{pmatrix}1\\ 1\end{pmatrix} = (A\cos t - B\sin t)\begin{pmatrix}1\\ 0\end{pmatrix} + (A\sin t + B\cos t)\begin{pmatrix}1\\ 1\end{pmatrix}\\ &= A\begin{pmatrix}\cos t + \sin t\\ \sin t\end{pmatrix} + B\begin{pmatrix}\cos t - \sin t\\ \cos t\end{pmatrix},\end{aligned}$$

and the initial conditions $(x(0), y(0)) = (0,1)$ correspond to $(A,B) = (-1,1)$:

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}-2\sin t\\ \cos t - \sin t\end{pmatrix}.$$

The origin is a centre, surrounded by closed trajectories with period 2π: cirles $u^2 + v^2 = C$ in the new coordinates, hence ellipses $(x-y)^2 + y^2 = C$ in the original coordinates. The particular solution curve here, the one passing through the point $(0,1)$, is the ellipse $(x-y)^2 + y^2 = 2$, traversed in the counterclockwise direction. The nullclines $\dot x = x - 2y = 0$ and $\dot y = x - y = 0$ are drawn in red and orange in the left picture below; note that the ellipse has a horizontal/vertical tangent precisely where it crosses the nullclines. (For a really precise description of the ellipse, one can diagonalize the quadratic form $(x-y)^2 + y^2 = x^2 - 2xy + 2y^2$ to find that the axes of the ellipse lie along the lines $2x + (\sqrt{5}-1)y = 0$ and $2x - (\sqrt{5}+1)y = 0$, as drawn in the right picture, but that's perhaps overkill here.)



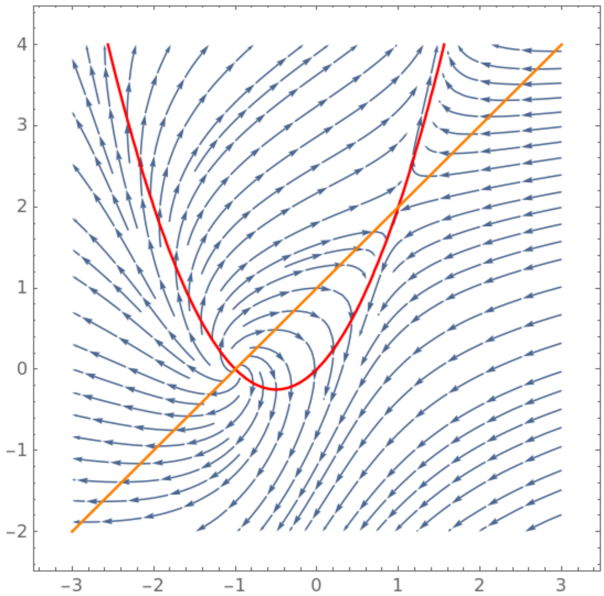
2. The nullclines $\dot{x} = 0 \iff y = x^2 + x = (x+\frac{1}{2})^2 - \frac{1}{4}$ (drawn in red in the figure below) and $\dot{y} = 0 \iff y = x + 1$ (drawn in orange) intersect at the equilibrium points $(x, y) = (-1, 0)$ and $(1, 2)$. The Jacobian is $J(x, y) = \left(\begin{smallmatrix} -2x-1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, so

$$J(-1,0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \beta = \operatorname{tr} J = 2 > 0, \qquad \gamma = \det J = 2 > (\beta/2)^2$$

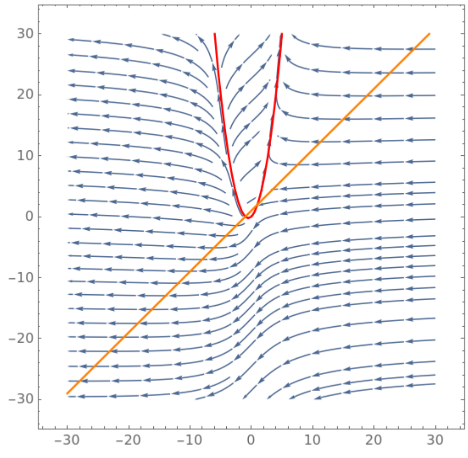
and

$$J(1,2) = \begin{pmatrix} -3 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \gamma = \det J = -2 < 0.$$

By the trace–determinant criterion, $(-1, 0)$ is an **unstable focus** and $(1, 2)$ is a **saddle** (where we can compute the principal directions $\left(\begin{smallmatrix} 2\pm\sqrt{3} \\ 1 \end{smallmatrix}\right)$ if we want to be really precise when drawing the phase portrait, but in this case we can draw a pretty good picture even without this information).



(In case you are interested in what the solution curves look like further away from the origin, note that far out to the left or to the right in the picture, i.e., if we consider large enough $|x|$ for each fixed y, the vector field is pointing nearly straight left, since the term $-x^2$ in $\dot{x}$ is overshadowing everything else. And far away on the y-axis, the slope of the curves is close to 45°, since if $x = 0$, then $\dot{y}/\dot{x} = (y-1)/y$, which tends to 1 as $y \to \pm\infty$.)



3. Inserting $x(t) = e^{kt}$ into the ODE gives

$$\begin{aligned} 0 &= (t^2+1)\,\ddot{x}(t) - 2(t^2+t+1)\,\dot{x}(t) + 4t\,x(t) \\ &= e^{kt}\Big(k^2(t^2+1) - 2k(t^2+t+1) + 4t\Big) \\ &= e^{kt}\Big((k^2-2k)(t^2+1) + (4-2k)\,t\Big), \end{aligned}$$

which is satisfied identically iff $k = 2$. So $x(t) = e^{2t}$ is a solution. To reduce the order, let $x(t) = e^{2t}\,Y(t)$; then $\dot{x} = e^{2t}(2Y + \dot{Y})$ and $\ddot{x} = e^{2t}(4Y + 4\dot{Y} + \ddot{Y})$, so the ODE becomes

$$\begin{aligned} 0 &= (t^2+1)\,\ddot{x}(t) - 2(t^2+t+1)\,\dot{x}(t) + 4t\,x(t) \\ &= e^{2t}\Big((t^2+1)(4Y + 4\dot{Y} + \ddot{Y}) - 2(t^2+t+1)(2Y + \dot{Y}) + 4t\,Y\Big) \\ &= e^{2t}\Big(0\,Y + 2(t^2-t+1)\dot{Y} + (t^2+1)\ddot{Y}\Big) \\ &= \frac{e^{2t}}{t^2+1}\left(\dot{y} + \left(2 - \frac{2t}{t^2+1}\right)y\right), \end{aligned}$$

where $y = \dot{Y}$. Then $\exp\big(2t - \ln(t^2+1)\big) = e^{2t}/(t^2+1)$ is an integrating factor, which turns the ODE into $\frac{d}{dt}\big(y(t)\,e^{2t}/(t^2+1)\big) = 0$, with the solution $y(t) = A(t^2+1)e^{-2t}$. Integration gives $Y(t) = \int y(t)\,dt = -\frac{1}{2}A(t^2+t+\frac{3}{2})e^{-2t} + B$. Letting $C = -A/2$ will make the final answer $x(t) = e^{2t}Y(t)$ look a little nicer.

Answer. $x(t) = Be^{2t} + C(t^2+t+\frac{3}{2})$, where B and C are arbitrary constants.

4. (a) Assume (for the moment) that $xy \neq 0$. Then $dy/dx = \dot{y}/\dot{x} = (1-x^2)/xy$, which gives

$$\int y\,dy = \int \frac{1-x^2}{x}\,dx \qquad \Longleftrightarrow \qquad \frac{y^2}{2} = \ln|x| - \frac{x^2}{2} + C,$$

so that $C = (x^2+y^2)/2 - \ln|x|$ is a constant of motion (for $x \neq 0$). If we want a constant of motion that is defined on all of $\mathbf{R}^2$ we can take e^{-C} and remove the absolute value signs to get $F(x,y) = x\,e^{-(x^2+y^2)/2}$. (Anyone who worries about the restriction $xy \neq 0$ can verify by direct computation that $\dot{F} = 0$ for all $(x,y) \in \mathbf{R}^2$.)

Answer. $F(x,y) = x\,e^{-(x^2+y^2)/2}$ (for example).

(b) A linear system has the form $\dot{x} = ax + by$, $\dot{y} = cx + dy$, and then

$$\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(ax+by) + 2y(cx+dy) = 2ax^2 + 2(b+c)xy + 2dy^2$$

is identically zero iff $a = d = 0$ and $c = -b$.

Answer. $\dot{x} = by$, $\dot{y} = -bx$, where $b \in \mathbf{R}$ is arbitrary.

(c) We can simply multiply some linear vector field from part (b), such as $(\dot{x}, \dot{y}) = (y, -x)$, by a nonzero and nonconstant function, such as $f(x, y) = e^x$, to get a nonlinear vector field with exactly the same phase portrait; it's only the speed along the trajectories that changes. (Actually it also works with a function f which has zeros; the phase portrait may gain some additional equilibrium points, but that doesn't change the fact that $\dot{V} = 0$.)

Answer. $\dot{x} = ye^x$, $\dot{y} = -xe^x$ (for example).

5. Obviously $V(x, y) = x^2 + y^2$ is positive definite, and $\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(-y + xy) + 2y(x - x^2 - y^3) = -2y^4 \le 0$ for all (x, y), so V is a weak Liapunov function for the system, and thus the origin is a stable equilibrium according to Liapunov's theorem.

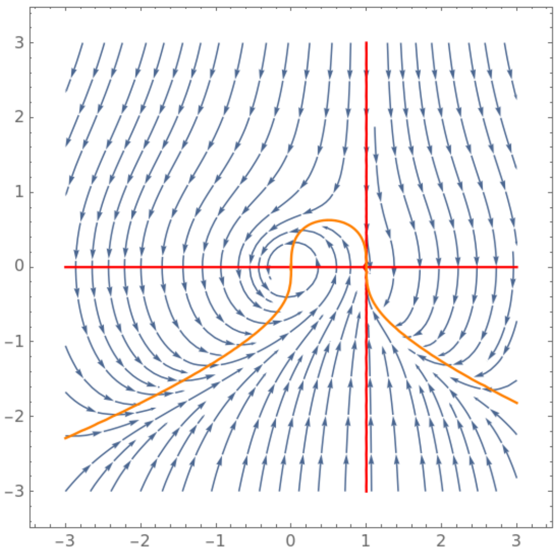
   It cannot be *globally* asymptotically stable, for the simple reason that $(x, y) = (1, 0)$ is another equilibrium point.

   However, the origin *is* asymptotically stable, by LaSalle's theorem applied with the restriction of V to the open half-plane $\Omega = \{(x, y) \in \mathbf{R}^2 : x < 1\}$. Indeed, V is of course still a weak Liapunov function when restricted to Ω, and the set

$$C = \left\{(x, y) \in \Omega : \dot{V} = 0\right\} = \left\{(x, 0) : x < 1\right\}$$

   contains no complete trajectories except the origin. (Proof: The vector field is transversal to the horizontal half-line C except at the origin, since if $y = 0$ and $x < 1$ and $x \neq 0$, then $\dot{y} = x - x^3 \neq 0$.) So the assumptions for LaSalle's theorem are satisfied.

   (Remark: In fact, all trajectories starting in Ω converge to the origin, and all other trajectories converge to $(1, 0)$. See the phase portrait below.)



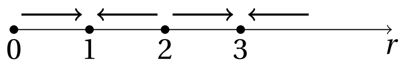
6. Let us look for a system of the form

$$\dot{x} = -y + x\,f(x^2 + y^2),$$
$$\dot{y} = x + y\,f(x^2 + y^2),$$

where f is a polynomial, since in polar coordinates this becomes

$$\dot{r} = r\,f(r^2), \qquad \dot{\theta} = 1,$$

which has the desired property if the one-dimensional phase portrait for r has stable equilibria at $r = 1$ and $r = 3$, perhaps like this:



For example, $\dot{r} = r(1-r^2)(4-r^2)(9-r^2)$ has such a phase portrait (for $r \geq 0$, which is all that is relevant here), so $f(t) = (1-t)(4-t)(9-t)$ will do.

Answer. For example,

$$\begin{aligned}\dot{x} &= -y + x(1-x^2-y^2)(4-x^2-y^2)(9-x^2-y^2),\\ \dot{y} &= x + y(1-x^2-y^2)(4-x^2-y^2)(9-x^2-y^2).\end{aligned}$$