Linköpings universitet
Matematiska institutionen
Hans Lundmark

Utbildningskod: TATA71
Modul: TEN1

# TATA71 Ordinary Differential Equations and Dynamical Systems

## Take-home examination 2021-03-16, 14.00–19.00

**Rules, short version.**

- Aids are permitted, but no collaboration with other persons is allowed.

**Rules, long version.**

- This is an individual examination, so you are required to answer the questions on your own.
- You may ask the teacher for clarifications (email **hans.lundmark@liu.se**). Except for that, it is not allowed to communicate in any way with other persons regarding the solutions of the problems during the exam. So you may not get help from others, and it is also not allowed to *give* help to other students who are taking this exam, for example by letting them look at your solutions.
- You can use any aids (books, computers, etc.), but you are expected to present your solutions with as much detail as if calculating by hand (like on a usual exam without aids). Consulting old information from online forums is allowed, but you may not make use of any questions or answers posted during the exam. Cite your sources whenever appropriate, and avoid quoting text verbatim; it is much preferred if you use your own formulations.
- The solutions should be handwritten (unless you have a special permit from LiU's disability coordinator to write on a computer). Writing by hand on a tablet is fine, but please use dark text on a white background.
- You may write your answers in English or in Swedish (or some mixture thereof).

You will find the problems **on the next page**.

Each problem will be marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least $n$ passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Draw the phase portrait for the equation $\dot{x} = x - x^3$, and derive an explicit formula for the solution $x(t)$ in terms of the initial value $x(0) = x_0 \in \mathbf{R}$. Together with this formula, also state the maximal interval of existence (depending on $x_0$).

2. Consider all linear systems

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy$$

such that $(x(t), y(t)) = (e^t, 2e^t)$ is a solution. Choose **two** such systems, of different algebraic types, and sketch their phase portraits. (As always, draw the nullclines and some typical solution curves, but also make sure to include the particular solution curve that was given above.)

3. In each part below, give (with explanation) an example of a system $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$ having the given property:

   (a) The origin is a saddle, with principal directions $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

   (b) There are equilibria at $(x, y) = (2, 4)$ and $(-1, 1)$ (and nowhere else).

   (c) The origin is the only equilibrium, and it is asymptotically stable but not globally asymptotically stable.

4. Use "variation of constants" to determine the general solution to

$$\ddot{x}(t) + \tfrac{t-1}{t}\,\dot{x}(t) - \tfrac{1}{t}\,x(t) = 2t\cos t, \qquad t > 0,$$

given that $e^{-t}$ and $t - 1$ satisfy the homogeneous equation.

5. Determine all pairs $(\lambda, \mu) \in \mathbf{R}^2$ such that the system

$$\dot{x} = \mu x + 6y, \qquad \dot{y} = -\mu x + (\mu - 2\lambda)y + \tfrac{1}{4}x^2$$

has a stable focus at the origin (as far as can be determined by linearization). Illustrate the result by drawing the set of all such pairs in the $(\lambda, \mu)$-plane. Then choose **one** such pair $(\lambda, \mu)$, and sketch the phase portrait of the corresponding system.

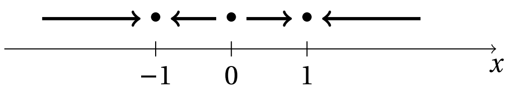
6. Find a constant of motion for the system

$$\dot{x} = y, \qquad \dot{y} = y^2 - x,$$

and sketch the phase portrait.

# Solutions for TATA71 2021-03-16

1. Phase portrait for $\dot{x} = x - x^3$:




The solution can be computed either by separation of variables $\int \frac{dx}{x-x^3} = \int dt$ (for $x \neq 0, \pm 1$) or by the substitution $y = 1/x^2$ (for $x \neq 0$); note that some care regarding signs, absolute values and exceptional cases is required either way.

**Answer.**

$$x(t) = \frac{x_0}{\sqrt{x_0^2 + (1 - x_0^2)e^{-2t}}},$$

defined for all $t \in \mathbf{R}$ if $-1 \leq x_0 \leq 1$, and for $t > \frac{1}{2}\ln(1 - x_0^{-2})$ otherwise.

2. The conditions for $(x(t), y(t)) = (e^t, 2e^t)$ to be a solution are $a + 2b = 1$ and $c + 2d = 2$. Of course the appearance of the phase portrait depends on your choice of values, but for example $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ gives a star node, while $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1/2 \\ 2 & 0 \end{smallmatrix}\right)$ gives a saddle with principal directions $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ (with eigenvalue $\lambda = 1$) and $\left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$ (with $\lambda = -1$).

3. **Answer.** For example:

   (a) $\dot{x} = 5x + 4y$, $\dot{y} = 6x - 5y$ (with eigenvalues 7 and $-7$, respectively).

   (b) $\dot{x} = y - x^2$, $\dot{y} = 2 + x - y$.

   (c) $\dot{x} = (x^2 + y^2 - 1)x - y$, $\dot{y} = (x^2 + y^2 - 1)y + x$ (which in polar coordinates is $\dot{r} = (r^2 - 1)r$, $\dot{\theta} = 1$).

4. Write the ODE as the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1/t & (1-t)/t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2t\cos t \end{pmatrix},$$

where $(x_1, x_2) = (x, \dot{x})$. A fundamental matrix is obtained from the given solutions to the homogeneous equation:

$$\Phi(t) = \begin{pmatrix} e^{-t} & t-1 \\ \frac{d}{dt}e^{-t} & \frac{d}{dt}(t-1) \end{pmatrix} = \begin{pmatrix} e^{-t} & t-1 \\ -e^{-t} & 1 \end{pmatrix}.$$

In the usual way, the change of variables $\mathbf{x}(t) = \Phi(t)\,\mathbf{y}(t)$ leads to

$$\dot{\mathbf{y}}(t) = \Phi(t)^{-1} \begin{pmatrix} 0 \\ 2t\cos t \end{pmatrix} = \frac{1}{te^{-t}} \begin{pmatrix} 1 & -(t-1) \\ e^{-t} & e^{-t} \end{pmatrix} \begin{pmatrix} 0 \\ 2t\cos t \end{pmatrix} = \begin{pmatrix} 2(1-t)e^t\cos t \\ 2\cos t \end{pmatrix}.$$

The simplest way of integrating $\dot{y}_1(t) = 2(1-t)e^t \cos t$ is probably

$$\begin{aligned} y_1(t) &= 2\,\mathrm{Re}\int e^{(1+i)t}(1-t)\,dt \\ &= 2\,\mathrm{Re}\left(\frac{e^{(1+i)t}}{1+i}(1-t) - \frac{e^{(1+i)t}}{(1+i)^2}(-1)\right) + C \\ &= e^t\Big((\cos t + \sin t)(1-t) + \sin t\Big) + C, \end{aligned}$$

and obviously $y_2(t) = 2\sin t + D$. Finally, we get $x(t) = x_1(t)$ from the first row of $\Phi(t)$ times the (now known) column vector $\mathbf{y}(t)$.

**Answer.** $x(t) = Ce^{-t} + D(t-1) + (1-t)\cos t + t\sin t$.

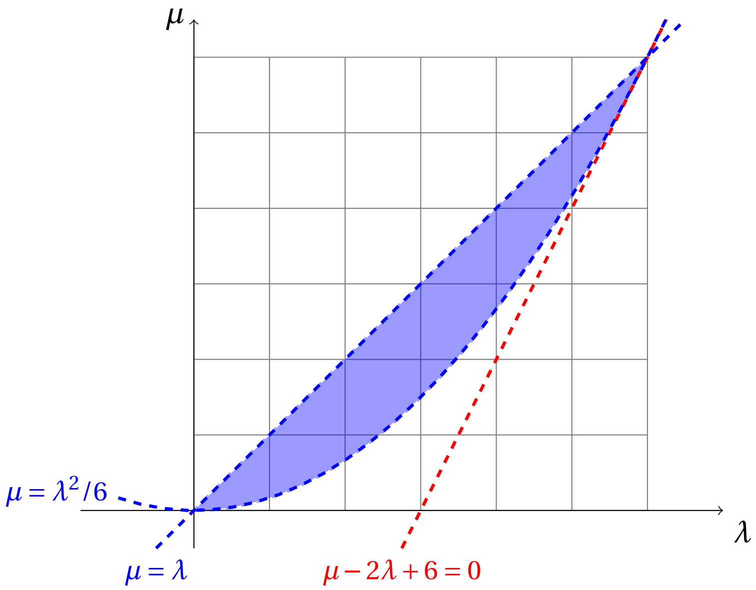
5. The linearization at the origin is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mu & 6 \\ -\mu & \mu - 2\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

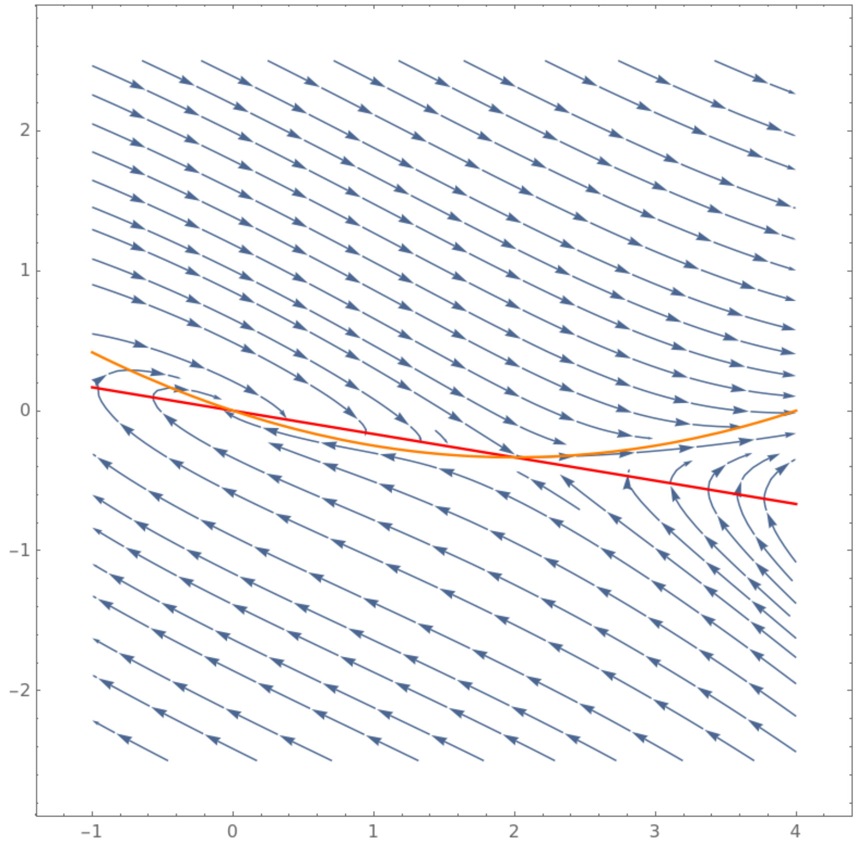
which is a stable focus iff the trace $\beta$ and the determinant $\gamma$ satisfy

$$\begin{aligned} &0 < -\beta = 2(\lambda - \mu), \\ &0 < \gamma = \mu(\mu - 2\lambda + 6), \\ &0 < \gamma - (\beta/2)^2 = 6\mu - \lambda^2. \end{aligned}$$

The first condition $\mu < \lambda$ and the third condition $\mu > \lambda^2/6$ determine a region where the second condition is automatically satisfied, since $\mu$ and $\mu - 2\lambda + 6$ are both positive there:



**Answer.** All pairs $(\lambda, \mu)$ such that $\lambda^2/6 < \mu < \lambda$ (the shaded region in the figure above). For example, with $(\lambda, \mu) = (2, 1)$ we get the following phase portrait, with a saddle at $(x, y) = (2, -\frac{1}{3})$:




6. Seeking $y$ as a function of $x$, we have (for $y \neq 0$)

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{y^2 - x}{y} = y - \frac{x}{y} \quad \Longleftrightarrow \quad \frac{dy}{dx} - y = -\frac{x}{y}.$$

Multiplication by the integrating factor $e^{-x}$ gives

$$\frac{d(ye^{-x})}{dx} = -\frac{xe^{-x}}{y}.$$

Let $z(x) = y(x)\,e^{-x}$. Then

$$\frac{dz}{dx} = -\frac{xe^{-2x}}{z},$$

which is a separable ODE:

$$\int z\,dz = -\int xe^{-2x}\,dx \quad \Longleftrightarrow \quad \frac{z^2}{2} = \frac{(2x+1)e^{-2x}}{4} + C.$$

Going back to $y$, and taking $D = 2C$ we find that

$$D(x, y) = (y^2 - x - \tfrac{1}{2})\, e^{-2x}$$

is a constant of motion. (The restriction $y \neq 0$ is irrelevant for the final result; $D$ is a global constant of motion, as is easily verified by computing $\dot{D} = 0$.) When drawing the phase portrait, notice in particular that the level curve $D = 0$ is just the parabola $x = y^2 - \frac{1}{2}$ (dashed in the figure below), and in general the trajectories follow the curves

$$y = \pm\sqrt{x + \tfrac{1}{2} + De^{2x}},$$

so that they are closed iff $D < 0$, i.e., to the right of the parabola $D = 0$.

**Answer.** Constant of motion: $(y^2 - x - \frac{1}{2})\, e^{-2x}$. Phase portrait: see below.