Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2021-08-27 kl. 8.00–13.00

No aids allowed. You may write your answers in English or Swedish (or both). Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points. Solutions will be posted on the course webpage afterwards. Good luck!

1. Derive an explicit formula for the solution $x(t)$ of the logistic equation
$$\dot{x} = rx\left(1 - \frac{x}{K}\right), \qquad r > 0, \quad K > 0,$$
in terms of the initial value $x(0) = x_0 \in \mathbf{R}$. Also state the maximal interval of existence for the solution (as a function of x_0).

2. Sketch the phase portrait for the linear system
$$\dot{x} = x + 3y, \qquad \dot{y} = -2y.$$
Try to make your drawing as accurate as possible. In particular, take the nullclines and the principal directions into account.

3. Suppose $(x(t), y(t))$ satisfies
$$\dot{x} = 2x + 3y, \qquad \dot{y} = -3x + 2y$$
with the initial values $(x(0), y(0)) = (0,1)$. Compute $(x(5), y(5))$.

4. Use linearization to classify the equilibria of the system
$$\dot{x} = y - x^2, \qquad \dot{y} = 2 + x - y,$$
and sketch the phase portrait.

5. Show that the origin is a stable equilibrium of the system
$$\dot{x} = -2y, \qquad \dot{y} = 2x + x^2 - y^3.$$
(Hint: $V(x,y) = x^2 + y^2 + \frac{1}{3}x^3$.) Is it asymptotically stable? If so, is it globally asymptotically stable?

6. Compute the general solution $x(t)$ of the ODE
$$\ddot{x} + \dot{x} - 2x = \frac{3}{e^t + e^{-t}}.$$

Solutions for TATA71 2021-08-27

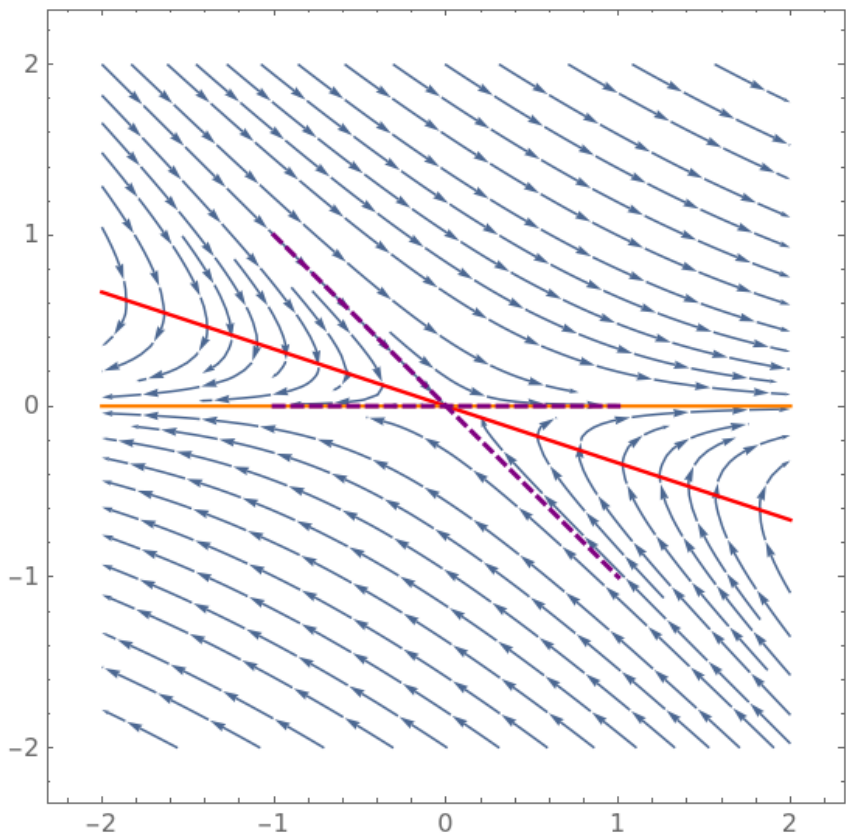
1. If $x_0 = 0$, then $x(t)$ is identically zero. Otherwise, let $y(t) = 1/x(t)$, in terms of which the ODE becomes $-\dot{y}/y^2 = (r/y)(1 - 1/(Ky))$, or in other words $\dot{y} + ry = r/K$, with the general solution $y(t) = Ae^{-rt} + 1/K$, where the initial condition $y(0) = 1/x_0$ gives $A = 1/x_0 - 1/K$. So for $x_0 \neq 0$ we have

$$x(t) = \frac{1}{y(t)} = \frac{1}{\left(\frac{1}{x_0} - \frac{1}{K}\right)e^{-rt} + \frac{1}{K}} = \frac{Kx_0}{x_0 + (K - x_0)e^{-rt}},$$

and this final expression also gives the correct solution when $x_0 = 0$, so it's valid for all $x_0 \in \mathbf{R}$. The solution ceases to exist if the denominator becomes zero, i.e., if $t = \frac{1}{r}\ln(1 - \frac{K}{x_0})$.

Answer. Formula for $x(t)$ as above. The maximal interval of existence is $\mathbf{R}$ if $0 \leq x_0 \leq K$, it's $\left(-\infty, \frac{1}{r}\ln(1 - \frac{K}{x_0})\right)$ if $x_0 < 0$, and it's $\left(\frac{1}{r}\ln(1 - \frac{K}{x_0}), \infty\right)$ if $x_0 > K$.

2. The origin is a saddle point, with principal directions $\binom{1}{0}$ (outgoing, eigenvalue 1) and $\binom{-1}{1}$ (incoming, eigenvalue -2). The x-nullcline is the line $y = -x/3$ (red), and the y-nullcline is the line $y = 0$ (yellow, happens to coincide with one of the principal directions).



3. The system is linear, and already in Jordan canonical form,

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}2 & 3\\ -3 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix},$$

with $\alpha = 2$ and $\beta = -3$, so we know that the general solution is

$$\begin{aligned}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} &= Ae^{\alpha t}\begin{pmatrix}\cos\beta t\\ \sin\beta t\end{pmatrix} + Be^{\alpha t}\begin{pmatrix}-\sin\beta t\\ \cos\beta t\end{pmatrix}\\ &= Ae^{2t}\begin{pmatrix}\cos 3t\\ -\sin 3t\end{pmatrix} + Be^{2t}\begin{pmatrix}\sin 3t\\ \cos 3t\end{pmatrix}.\end{aligned}$$

From the initial data $(x(0), y(0)) = (0, 1)$ we find $A = 0$ and $B = 1$, so that

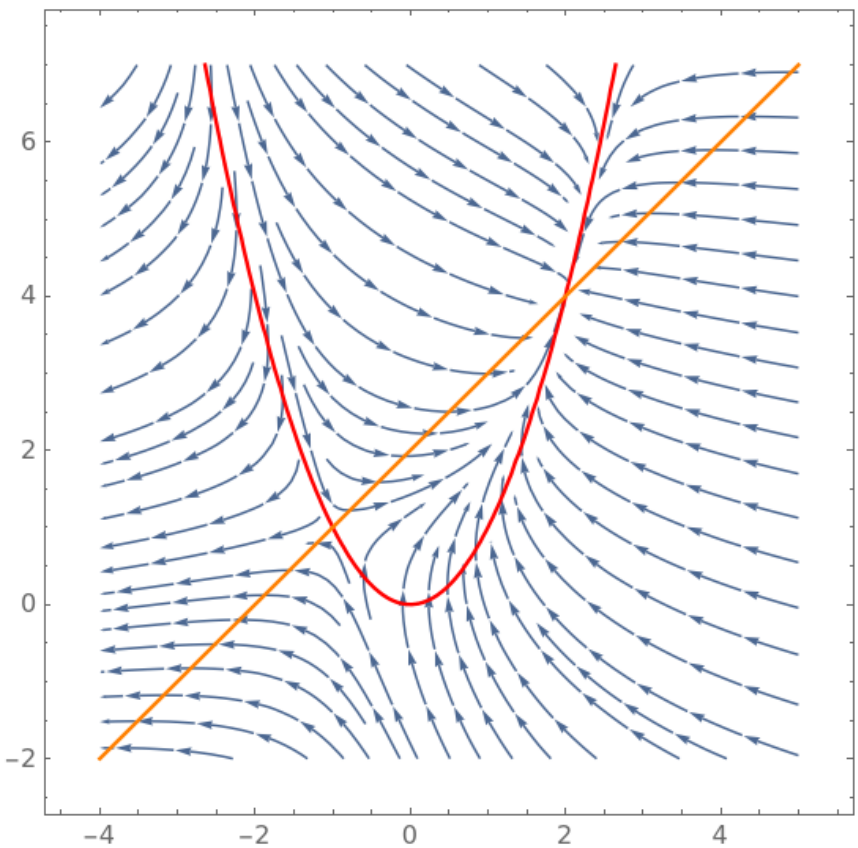
$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = e^{2t}\begin{pmatrix}\sin 3t\\ \cos 3t\end{pmatrix}.$$

Answer. $(x(5), y(5)) = (e^{10}\sin 15, e^{10}\cos 15)$.

4. There are two equilibria, $(x, y) = (-1, 1)$ and $(x, y) = (2, 4)$. Jacobian matrix:

$$J(x,y) = \begin{pmatrix}-2x & 1\\ 1 & -1\end{pmatrix} \quad\Longrightarrow\quad J(-1,1) = \begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}, \quad J(2,4) = \begin{pmatrix}-4 & 1\\ 1 & -1\end{pmatrix}.$$

The trace-determinant criterion shows that $(-1, 1)$ is a **saddle point**, since $\gamma = \det J(-1,1) = -3$ is negative, while $(2, 4)$ is a **stable node**, since $\beta = \operatorname{tr} J(2,4) = -5$ is negative, $\gamma = \det J(2,4) = 3$ is positive, and $\gamma < (\beta/2)^2$.



5. We have $V(0,0) = 0$ and $V(x,y) = x^2(1+\frac{1}{3}x) + y^2 > 0$ if $(x,y) \neq (0,0)$ and $x > -3$, and moreover

$$\dot{V} = (2x+x^2)\dot{x} + 2y\dot{y} = -2y(2x+x^2) + 2y(2x+x^2-y^3) = -2y^4 \leq 0$$

for all (x,y), so that V is a weak Liapunov function in the region $\Omega_1 = \{(x,y) : x > -3\}$. Hence, by Liapunov's theorem, the origin is a **stable** equilibrium. Next, consider the subregion $\Omega_2 = \{(x,y) : x > -2\}$, where V is obviously still a weak Liapunov function. The set of points in Ω where $\dot{V} = 0$, call it S, is the portion of the x-axis where $x > -2$. For $(0,0) \neq (x,y) \in S$, we have $\dot{x} = 0$ and $\dot{y} = 2x + x^2 = x(x+2) \neq 0$, so that the vector field is transversal to S except at the origin; hence the only complete trajectory contained in S is the origin, so the hypotheses for LaSalle's theorem are fulfilled, showing that the origin is in fact **asymptotically stable**. But it is **not globally asymptotically stable**, for the simple reason that there is another equilibrium point $(-2,0)$.

6. The homogeneous equation $\ddot{x} + \dot{x} - 2x = (D-1)(D+2)x = 0$ is equivalent to $x(t) = Ae^t + Be^{-2t}$, so by variation of constants the solution is $x(t) = a(t)e^t + b(t)e^{-2t}$ where

$$e^t\dot{a}(t) + e^{-2t}\dot{b}(t) = 0, \qquad e^t\dot{a}(t) - 2e^{-2t}\dot{b}(t) = \frac{3}{e^t + e^{-t}}.$$

This gives

$$\dot{a}(t) = \frac{e^{-t}}{e^t + e^{-t}}, \qquad \dot{b}(t) = \frac{-e^{2t}}{e^t + e^{-t}},$$

so that (with $u = e^{-t}$)

$$\begin{aligned} a(t) &= \int \frac{e^{-t}\,dt}{e^t + e^{-t}} = \int \frac{-du}{u^{-1} + u} = -\int \frac{u\,du}{1+u^2} \\ &= -\tfrac{1}{2}\ln(1+u^2) + A = -\tfrac{1}{2}\ln(1+e^{-2t}) + A \end{aligned}$$

and (with $v = e^t$)

$$\begin{aligned} b(t) &= -\int \frac{e^{2t}\,dt}{e^t + e^{-t}} = -\int \frac{v\,dv}{v + v^{-1}} = -\int \left(1 - \frac{1}{1+v^2}\right) dv \\ &= -(v - \arctan v) + B = \arctan(e^t) - e^t + B \end{aligned}$$

Answer. $x(t) = Ae^t + Be^{-2t} - \frac{1}{2}e^t\ln(1+e^{-2t}) + e^{-2t}\arctan(e^t) - e^{-t}$.