Linköpings universitet
Matematiska institutionen
Hans Lundmark

Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2022-01-13 kl. 8.00–13.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Compute the general solution of the linear system $\dot{x} = 3x$, $\dot{y} = -2x - y$, and draw the phase portrait as carefully as you can.

2. Consider the following model for a fish population of size $x(t)$:

$$\frac{dx}{dt} = \underbrace{rx\left(1 - \frac{x}{K}\right)}_{\text{logistic growth}} - \underbrace{\frac{ax}{x+b}}_{\text{fishing}} \qquad (r, K, a, b > 0).$$

   (a) If x is measured in units of mass (and t is time, of course), what are the units of the parameters r, K, a and b?

   (b) Show that the variables can be rescaled to bring the ODE to the dimensionless form

$$\frac{dX}{d\tau} = X(1 - X) - \frac{\alpha X}{X + \beta}.$$

   State clearly how the new variables (X, τ) are related to (x, t), and how the new parameters (α, β) are related to (r, K, a, b).

3. Investigate stability of equilibria, and sketch the phase portrait, for the system $\dot{x} = x(y-1)$, $\dot{y} = y - x^3$.

4. (a) Find a constant of motion $H(x, y)$ for the system $\dot{x} = y$, $\dot{y} = -x + x^3$.

   (b) Show that the origin is an asymptotically stable equilibrium for the system $\dot{x} = y - x^3$, $\dot{y} = -x + x^3$. (Hint: Part (a) may be useful.)

5. Use variation of constants to determine the general solution of the ODE

$$\ddot{x}(t) - 4\dot{x}(t) + 3x(t) = \frac{e^{2t}}{1 + e^t}.$$

6. Suppose that $\beta = 1/2$ in the ODE for $X(\tau)$ from problem 2(b). Draw the phase portrait for $X \geq 0$.

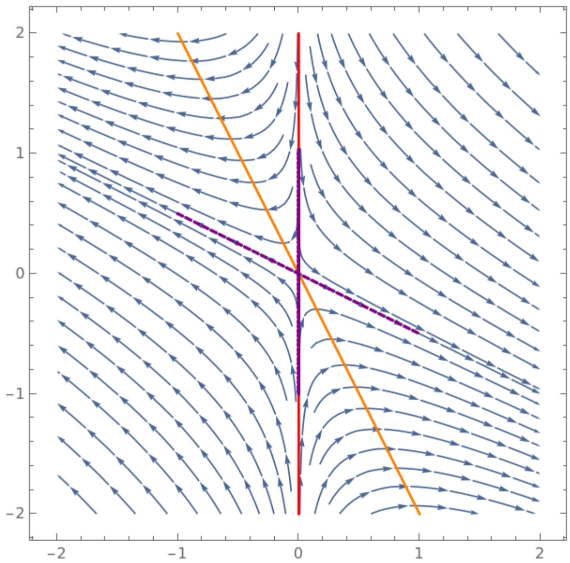
   (There will be several qualitatively different cases, depending on the value of $\alpha > 0$. State clearly what these cases are, and draw a separate phase portrait for each case.)

Solutions for TATA71 2022-01-13

1. The system matrix $\begin{pmatrix} 3 & 0 \\ -2 & -1 \end{pmatrix}$ has eigenvalues 3 and -1 with eigenvectors $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively, so the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with arbitrary constants $C_1, C_2 \in \mathbf{R}$. The phase portrait is a saddle, with straight-line trajectories along the eigenvectors (the principal directions); the other trajectories are hyperbolas with those lines as asymptotes. Taking also the nullclines $x = 0$ (red) and $y = -2x$ (orange) into account, we get the following picture:



2. (a) The unit of r is 1/time, for K and b it is mass, and for a it is mass/time.

   (b) With $x = KX$ and $t = \tau / r$ we get

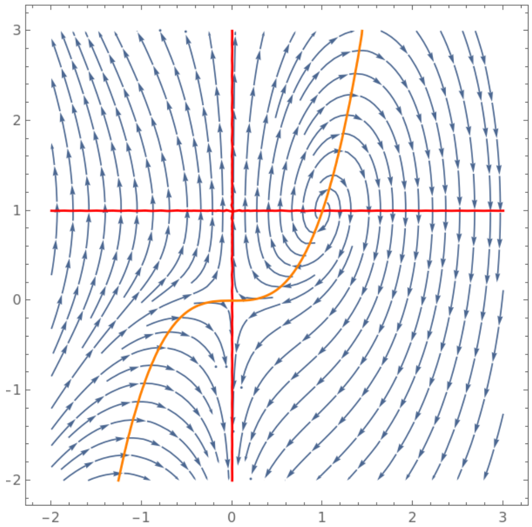
$$\frac{dX}{d\tau} = \frac{1}{Kr}\frac{dx}{dt} = \frac{1}{Kr}\left(rKX\left(1 - \frac{KX}{K}\right) - \frac{aKX}{KX+b}\right) = X(1-X) - \frac{\frac{a}{Kr}\cdot X}{X + \frac{b}{K}},$$

which is of the desired form, with $\alpha = \frac{a}{Kr}$ and $\beta = \frac{b}{K}$. (As a sanity check we may note, using the answer from part (a), that X, τ, α and β are indeed dimensionless.)

3. The equilibria are easily computed to be $(0,0)$ and $(1,1)$. The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} y-1 & x \\ -3x^2 & 1 \end{pmatrix}, \qquad J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J(1,1) = \begin{pmatrix} 0 & 1 \\ -3 & 1 \end{pmatrix},$$

so $(0,0)$ is obviously a saddle point (with eigenvalues ± 1 and with the x and y directions as principal directions), and hence unstable, while $(1,1)$ is an unstable focus, since $\beta = \operatorname{tr} J = 1 > 0$, $\gamma = \det J = 3 > 0$ and $\gamma - (\beta/2)^2 = 11/4 > 0$. Taking the nullclines and the direction of the vector field into account, we can now sketch the phase portrait:



4. (a) One may recognize that the system is Hamiltonian, $\dot{x} = \partial H/\partial y$, $\dot{y} = -\partial H/\partial x$, with $H(x,y) = \frac{1}{2}(x^2+y^2) - \frac{1}{4}x^4$, and then H is automatically a constant of motion. It's also possible to find H via $dy/dx = \dot{y}/\dot{x} = (-x+x^3)/y$, etc.

   (b) The function $H(x,y) = \frac{1}{2}x^2\left(1-\frac{1}{2}x^2\right) + \frac{1}{2}y^2$ from part (a) is a weak Liapunov function for the system in the strip $\Omega = \{(x,y) : -1 < x < 1\}$, since $H(0,0) = 0$, $H(x,y) > 0$ for $(x,y) \in \Omega \setminus \{(0,0)\}$ and

$$\dot{H} = \frac{\partial H}{\partial x}\,\dot{x} + \frac{\partial H}{\partial y}\,\dot{y} = (x-x^3)(y-x^3) + y(-x+x^3) = -x^4(1-x^2) \le 0$$

   for $(x,y) \in \Omega$. Moreover, the set of points in Ω where $\dot{H} = 0$, i.e., the line $x = 0$, contains no complete trajectories except for the equilibrium at the origin, since $(\dot{x},\dot{y}) = (y,0)$ when $x = 0$ so that the trajectories cross the line $x = 0$ transversally. Thus the hypotheses for LaSalle's theorem are satisfied, and this implies that the origin is asymptotically stable, as was to be shown.

5. Let $x_1 = x$ and $x_2 = \dot{x}$ to obtain the equivalent first-order system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^{2t}/(1+e^t) \end{pmatrix}.$$

We need a fundamental matrix for the homogeneous system, and the quickest way is perhaps to note that the general solution of $\ddot{x}(t) - 4\dot{x}(t) + 3x(t) = 0$ is $x(t) = Ae^t + Be^{3t}$, so that

$$\Phi(t) = \begin{pmatrix} e^t & e^{3t} \\ \frac{d}{dt}e^t & \frac{d}{dt}e^{3t} \end{pmatrix} = \begin{pmatrix} e^t & e^{3t} \\ e^t & 3e^{3t} \end{pmatrix}$$

works. Now we can let $\mathbf{x}(t) = \Phi(t)\,\mathbf{y}(t)$ and compute

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \Phi(t)^{-1} \begin{pmatrix} 0 \\ \frac{e^{2t}}{1+e^t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^{-3t} & e^{-3t} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{e^{2t}}{1+e^t} \end{pmatrix} = \frac{1}{2(1+e^t)} \begin{pmatrix} -e^t \\ e^{-t} \end{pmatrix},$$

which can be integrated using (for example) the substitution $u = e^t > 0$:

$$y_1(t) = -\frac{1}{2} \int \frac{e^t\,dt}{1+e^t} = -\frac{1}{2} \int \frac{du}{1+u} = -\frac{1}{2} \ln|1+u| + A = -\frac{1}{2} \ln(1+e^t) + A$$

and

$$\begin{aligned} y_2(t) &= \frac{1}{2} \int \frac{e^{-t}\,dt}{1+e^t} = \frac{1}{2} \int \frac{du}{u^2(1+u)} = \frac{1}{2} \int \left(\frac{1}{u^2} - \frac{1}{u} + \frac{1}{1+u} \right) du \\ &= \frac{1}{2} \left(-\frac{1}{u} - \ln|u| + \ln|1+u| \right) + B = \frac{1}{2} \big(-e^{-t} - t + \ln(1+e^t) \big) + B. \end{aligned}$$

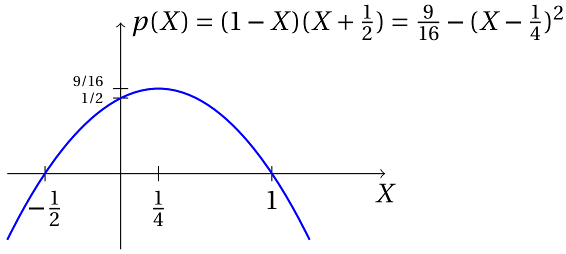
So the general solution $x(t) = x_1(t) = y_1(t)\,e^t + y_2(t)\,e^{3t}$ is

$$x(t) = Ae^t + Be^{3t} - \tfrac{1}{2} \Big(e^{2t} + t\,e^{3t} + \big(e^t - e^{3t}\big) \ln(1+e^t) \Big).$$

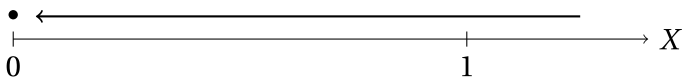
6. By completing the square, write the ODE as

$$\frac{dX}{d\tau} = X(1-X) - \frac{aX}{X+\frac{1}{2}} = \frac{X\big((1-X)(X+\frac{1}{2}) - \alpha\big)}{X+\frac{1}{2}} = \frac{X\big(\overbrace{\tfrac{9}{16} - (X-\tfrac{1}{4})^2}^{=p(X)} - \alpha\big)}{X+\frac{1}{2}}.$$

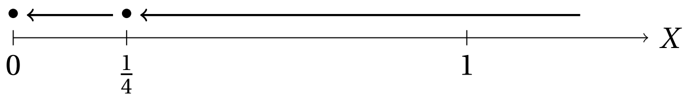
There's always an equilibrium at $X = 0$, and the rest of the phase portrait is determined by the sign of the factor $p(X) - \alpha$ for $X > 0$. The graph of $p(X)$ is a parabola, increasing from $p(0) = 1/2$ to the highest value $p(1/2) = 9/16$ and then decreasing down to $p(1) = 0$:



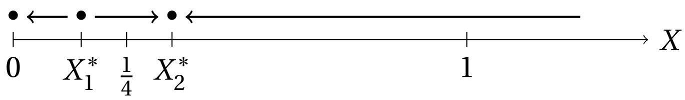
So if $\alpha > 9/16$, then $p(X) - \alpha$ is negative for all $X > 0$, and the phase portrait takes the following simple form (the fish population goes extinct due to overfishing; note that large values of $\alpha = \frac{a}{Kr}$ correspond to large values of the maximal fishing rate a):



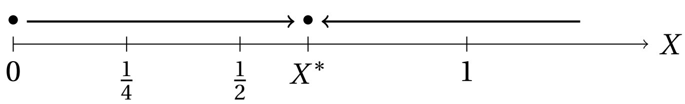
If $\alpha = 9/16$, then $p(X) - \alpha$ is negative for all $X > 0$ except that it's zero for $X = 1/4$:



If $1/2 < \alpha < 9/16$, then $p(X) - \alpha$ is zero for two positive values X_1^* and X_2^* (symmetrically located around $X = 1/4$), positive in between, and negative otherwise:



And if $0 < \alpha \le 1/2$, then $p(X) - \alpha$ is zero for one positive value $X^* \in \left[\frac{1}{2}, 1\right)$, positive for $0 < X < X^*$, and negative for $X^* < X$:



(If you have nothing better to do, and would like a little challenge, you could try to classify the phase portrait for all $\alpha > 0$ and $\beta > 0$ by studying the sign of $p(X) - \alpha = (1-X)(X+\beta) - \alpha$ for $X > 0$. As long as $0 < \beta < 1$, you will get similar results as for $\beta = 1/2$ above, with $\beta < \alpha < \frac{1}{4}(\beta+1)^2$ instead of $\frac{1}{2} < \alpha < \frac{9}{16}$, and so on. But if $\beta \ge 1$, then the case with X_1^* and X_2^* cannot occur, and the cases that need to be distinguished are $0 < \alpha < \beta$ and $\alpha \ge \beta$.)