Linköpings universitet
Matematiska institutionen
Hans Lundmark

Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2022-03-17 kl. 14.00–19.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Draw the phase portrait for the equation $\dot{x} = x^2 - 5x + 6$, and compute the solution satisfying the initial condition $x(0) = 1$. What is the maximal time interval where that solution is defined?

2. Compute the general solution of the linear system

$$\dot{x} = x - 4y, \qquad \dot{y} = 2x - 3y,$$

and draw the phase portrait.

3. For the system

$$\dot{x} = y - x^3, \qquad \dot{y} = 2(x - y^2),$$

investigate stability of equilibrium points and draw the phase portrait.

4. Find the general solution of the ODE $t^2\ddot{x} - 2t\dot{x} + 2x = 0$, for example by looking for solutions of the form $x(t) = t^n$. Then use variation of constants to solve $t^2\ddot{x} - 2t\dot{x} + 2x = t^3e^t$.

5. Use LaSalle's theorem with $V(x,y) = x^2 + y^2$ to show that the origin is an asymptotically stable equilibrium for the system

$$\dot{x} = -y - x^3 + 2x^4, \qquad \dot{y} = x - x^2y.$$

Also determine a region of stability.

6. Let $f(x,y) = 1 - (x^2 + y^2)(1 + x^2)$. Show that the system

$$\dot{x} = -y + x\,f(x,y), \qquad \dot{y} = x + y\,f(x,y)$$

has at least one limit cycle.

(Hint: Polar coordinates and the Poincaré–Bendixson theorem.)

Solutions for TATA71 2022-03-17

1. Since $\dot{x} = (x-2)(x-3)$, the phase portrait looks like "$\longrightarrow 2 \longleftarrow 3 \longrightarrow$".

   The solution $x(t)$ with $x(0) = 1$ stays in the interval $x < 2$, so

   $$t = \int_0^t dt = \int_1^{x(t)} \frac{dx}{(x-2)(x-3)} = \ln\left|\frac{x(t)-3}{x(t)-2}\right| - \ln\left|\frac{1-3}{1-2}\right| = \ln\frac{3-x(t)}{2(2-x(t))},$$

   where we can solve for

   $$x(t) = \frac{4e^t-3}{2e^t-1}.$$

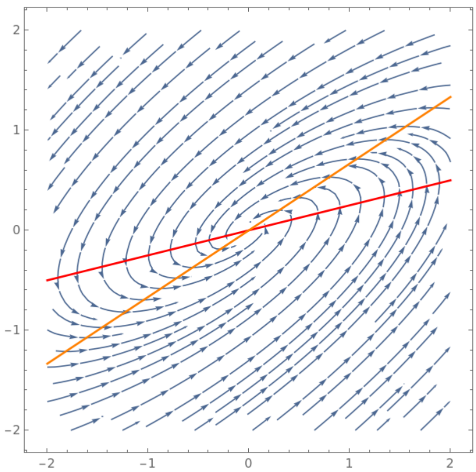
   For $t \geq 0$ there are no problems, but as we go backwards in time we encounter a singularity when $2e^t - 1$ becomes zero at time $t = \ln\frac{1}{2} = -\ln 2$, and then the solution blows up to $-\infty$ and ceases to exist. So the maximal interval of existence is $]-\ln 2, \infty[$.

   Answer. $x(t) = \dfrac{4e^t-3}{2e^t-1}$, for $t > -\ln 2$.

2. The system can be written as

   $$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 & -4\\ 2 & -3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix},$$

   and it's a stable focus since $\beta = \operatorname{tr} A = -2 < 0$ and $\gamma = \det A = 5 > (\beta/2)^2$. Taking the nullclines $\dot{x} = 0$ (red) and $\dot{y} = 0$ (orange) into account, we can draw the phase portrait:

   There are many ways of computing the general solution; for example, elimination of y gives $\ddot{x} + 2\dot{x} + 5x = 0$, which can be solved using the characteristic polynomial to give $x(t) = e^{-t}(A\cos 2t + B\sin 2t)$, and then $y = \frac{1}{4}(x - \dot{x})$ gives $y(t) = \frac{1}{2}e^{-t}(A\cos 2t + B\sin 2t) - \frac{1}{2}e^{-t}(-A\sin 2t + B\cos 2t)$.

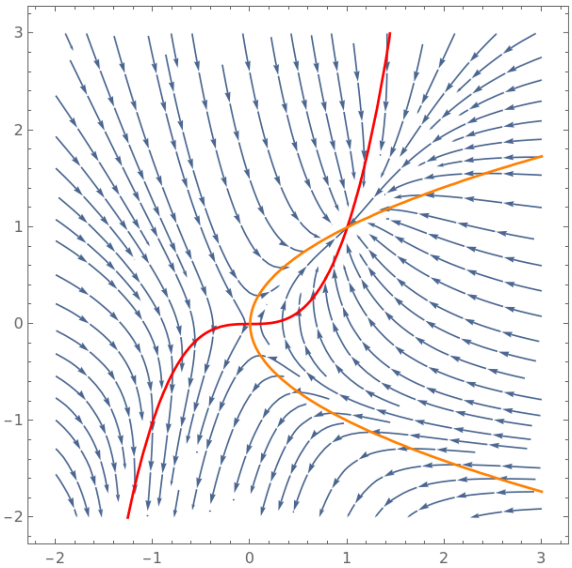
Answer. $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Ae^{-t}\begin{pmatrix} \cos 2t \\ \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t \end{pmatrix} + Be^{-t}\begin{pmatrix} \sin 2t \\ \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t \end{pmatrix}$.

3. The nullclines $y = x^3$ and $x = y^2$ intersect at the equilibrium points $(x, y) = (0,0)$ and $(1,1)$. The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} -3x^2 & 1 \\ 2 & -4y \end{pmatrix}, \quad J(0,0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad J(1,1) = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix},$$

so $(0,0)$ is a **saddle** (eigenvalues $\pm\sqrt{2}$, principal directions $(1, \pm\sqrt{2})$) and hence unstable, while $(1,1)$ is a **stable node** (eigenvalues -5 and -2, principal directions $(-1,2)$ and $(1,1)$).

Phase portrait:



4. Plugging $x(t) = t^n$ into $t^2\ddot{x} - 2t\dot{x} + 2x = 0$ yields $t^2 \cdot n(n-1)t^{n-2} - 2t \cdot nt^{n-1} + 2t^n = 0$, or in other words $n(n-1) - 2n + 2 = 0$, so that $n = 1$ or $n = 2$. Thus we find two linearly independent solutions $x(t) = t$ and $x(t) = t^2$, which means that the general solution to the homogeneous equation is $x(t) = At + Bt^2$.

The inhomogeneous equation $t^2\ddot{x} - 2t\dot{x} + 2x = t^3e^t$ is satisfied at $t = 0$ as long as $\dot{x}(0)$ and $\ddot{x}(0)$ exist and $x(0) = 0$. For $t \neq 0$, divide the equation by t^2 and let $x_1 = x$ and $x_2 = \dot{x}$ in order to write it as a system,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2/t^2 & 2/t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ te^t \end{pmatrix}.$$

From $x(t) = At + Bt^2$ and $\dot{x}(t) = A \cdot 1 + B \cdot 2t$ we obtain the fundamental matrix

$$\Phi(t) = \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix},$$

and now we let $\mathbf{x} = \Phi \mathbf{y}$ as usual, which leads to

$$\Phi(t)\,\dot{\mathbf{y}}(t) = \begin{pmatrix} 0 \\ te^t \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \frac{1}{t^2} \begin{pmatrix} 2t & -t^2 \\ -1 & t \end{pmatrix} \begin{pmatrix} 0 \\ te^t \end{pmatrix} = \begin{pmatrix} -te^t \\ e^t \end{pmatrix}.$$

Integration gives $y_1(t) = -(t-1)e^t + A$ and $y_2(t) = e^t + B$, so $x(t) = x_1(t) = t y_1(t) + t^2 y_2(t) = te^t + At + Bt^2$.

(At least that's what we get for $t \neq 0$, possibly with different constants (A, B) in the intervals $t > 0$ and $t < 0$. But if we want the ODE to be satisfied at $t = 0$, then we must take $x(0) = 0$ and use the same (A, B) for $t < 0$ and $t > 0$, so that $\dot{x}(0)$ and $\ddot{x}(0)$ exist.)

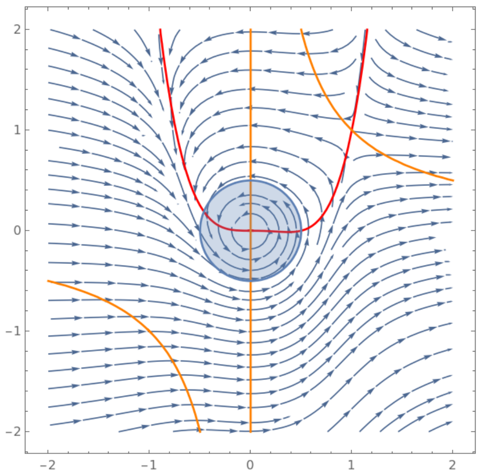
Answer. $x(t) = te^t + At + Bt^2$, for $t \in \mathbf{R}$.

5. With $V(x, y) = x^2 + y^2$, which is positive definite, we find

$$\tfrac{1}{2}\dot{V} = x\dot{x} + y\dot{y} = x(-y - x^3 + 2x^4) + y(x - x^2 y) = -x^4(1 - 2x) - x^2 y^2,$$

so $\dot{V} \leq 0$ in the open half-plane $\Omega = \{(x, y) : x < 1/2\}$, and thus V is a weak Liapunov function there. Inside Ω we have $\dot{V}(x, y) = 0$ if and only if (x, y) lies on the line $x = 0$, where $\dot{x} = -y \neq 0$ away from the origin; thus, that line contains no complete trajectories except the equilibrium point at the origin. By LaSalle's theorem, the origin is therefore asymptotically stable. The usual recipe (take a closed disk $x^2 + y^2 \leq r^2$, with $0 < r < \frac{1}{2}$ so that it lies in Ω, find the minimum of V on the ball's boundary, etc.) shows that every open disk $x^2 + y^2 < r^2$ with $0 < r < \frac{1}{2}$ is a domain of stability, and consequently so is the union of all of them, the open disk $x^2 + y^2 < \frac{1}{4}$.

(Our answer $x^2 + y^2 < \frac{1}{4}$ is far from optimal, as the computer-drawn phase portrait below shows, but it's the best that we get from this particular Liapunov function.)



6. Away from the origin, we have in polar coordinates

$$\begin{aligned}\dot r &= \frac{x\dot x + y\dot y}{r} = \frac{x\big(-y + x\,f(x,y)\big) + y\big(x + y\,f(x,y)\big)}{r}\\ &= \frac{(x^2+y^2)\,f(x,y)}{r} = r\big(1 - r^2(1 + r^2\cos^2\theta)\big)\end{aligned}$$

and

$$\dot\theta = \frac{x\dot y - y\dot x}{r^2} = \frac{x\big(x + y\,f(x,y)\big) - y\big(-y + x\,f(x,y)\big)}{r^2} = \frac{x^2+y^2}{r^2} = 1.$$

So, for example, if $r = \frac{1}{2}$ then

$$\dot r = \tfrac{1}{2}\big(1 - \tfrac{1}{4}\cdot(1 + \tfrac{1}{4}\cos^2\theta)\big) \geq \tfrac{1}{2}\big(1 - \tfrac{1}{4}\cdot\tfrac{5}{4}\big) > 0$$

for all θ, and if $r = 2$ then

$$\dot r = 2\big(1 - 4\cdot(1 + 4\cos^2\theta)\big) \leq 2\big(1 - 4\cdot 1\big) < 0$$

for all θ. This implies that the annulus $\frac{1}{2} \leq r \leq 2$ is a trapping region, and it contains no equilibrium (since $\dot\theta = 1$), so by the Poincaré–Bendixson theorem it must contain at least one limit cycle.