Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2023-01-12 kl. 8.00–13.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Compute the general solution and draw the phase portrait for the linear system $\dot{x} = 2x + 4y$, $\dot{y} = x - y$.

2. Investigate stability of the equilibrium points and sketch the phase portrait for the system $\dot{x} = -y$, $\dot{y} = 2x - x^2 - y$.

3. Given that $x(t) = t$ is a solution, use reduction of order to find the general solution of the ODE

$$(1+t^2)\ddot{x}(t) - 2t\dot{x}(t) + 2x(t) = 0.$$

4. Consider the following two-species population model, where p, r, s, K and L are positive parameters:

$$dx/dt = rx(1 - x/K), \qquad dy/dt = sy\big(1 - y/L\big) - pxy,$$

   Suggest a brief interpretation of what kind of situation this system may describe. Show how rescale the variables (x, y, t) to obtain the dimensionless system

$$dX/d\tau = X(1-X), \qquad dY/d\tau = \alpha Y(1-Y) - \beta XY.$$

   What condition on α and β is required for the existence of a nontrivial equilibrium (X^*, Y^*) with $X^* > 0$ and $Y^* > 0$?

5. Let $\mathbf{x}^*$ be an equilibrium point for the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^n$. Give the precise definitions of the following concepts:

   (a) $\mathbf{x}^*$ is **stable**.

   (b) $\mathbf{x}^*$ is **asymptotically stable**.

   (c) $\mathbf{x}^*$ is **globally asymptotically stable**.

6. Show that the origin is an asymptotically stable equilibrium for the system

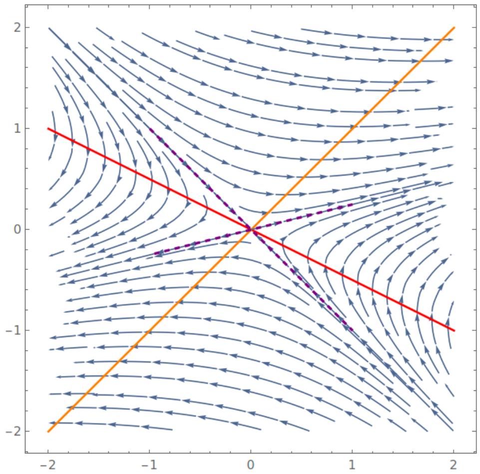
$$\dot{x} = -y + \left(x - \tfrac{3}{4}\right) y^4, \qquad \dot{y} = 4x - y^3,$$

   and determine a domain of stability.

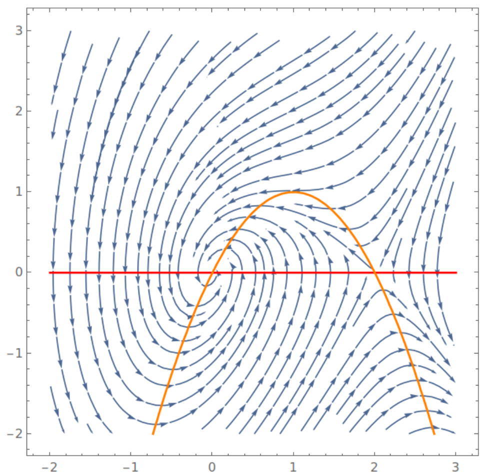
   (Hint: Try a Liapunov function of the form $V(x,y) = ax^2 + y^2$.)

Solutions for TATA71 2023-01-12

1. The system's matrix $\begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$ has eigenvalues 3 and -2 with eigenvectors $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, respectively, so the general solution is
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Ae^{3t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix},$$
where A and B are arbitrary real constants. The phase portrait is a saddle; taking the nullclines $x + 2y = 0$ and $x - y = 0$ into account, together with the principal directions found above, we obtain the following picture:



2. The nullclines $y = 0$ and $y = x(2 - x)$ intersect at the equilibrium points $(x, y) = (0, 0)$ and $(2, 0)$. The Jacobian matrix is
$$J(x, y) = \begin{pmatrix} 0 & -1 \\ 2 - 2x & -1 \end{pmatrix}, \quad J(0, 0) = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}, \quad J(2, 0) = \begin{pmatrix} 0 & -1 \\ -2 & -1 \end{pmatrix}.$$
The trace–determinant criterion shows that $(0, 0)$ is a **stable focus** since $\operatorname{tr} J(0, 0) = -1 < 0$ and $\det J(0, 0) = 2 > \left(\frac{1}{2} \operatorname{tr} J(0, 0)\right)^2 = \frac{1}{4}$, while $(2, 0)$ is a **saddle** since $\det J(2, 0) = -2 < 0$; we may also compute the eigenvalues -2 and 1 with corresponding principal directions $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Phase portrait:



3. Let $x(t) = t\,Y(t)$. Then $\dot{x} = Y + t\dot{Y}$ and $\ddot{x} = 2\dot{Y} + t\ddot{Y}$, which inserted into the ODE gives $0 = (1+t^2)\ddot{x} - 2t\dot{x} + 2x = (1+t^2)(2\dot{Y} + t\ddot{Y}) - 2t(Y + t\dot{Y}) + 2tY = 0Y + 2\dot{Y} + t(1+t^2)\ddot{Y} = 2y + t(1+t^2)\dot{y}$, where $y(t) = \dot{Y}(t)$. Writing this as

$$0 = \dot{y} + \frac{2}{t(1+t^2)}y = \dot{y} + \left(\frac{2}{t} - \frac{2t}{1+t^2}\right)y,$$

we find the integrating factor $\exp\bigl(2\ln|t| - \ln(1+t^2)\bigr) = t^2/(1+t^2)$, so that $\frac{t^2}{1+t^2}\,y(t) = C$, and we obtain the general solution $x(t) = tY(t)$ where

$$Y(t) = \int y(t)\,dt = C\int \frac{1+t^2}{t^2}\,dt = C\left(-\frac{1}{t} + t\right) + D.$$

Answer. $x(t) = C(t^2 - 1) + Dt$, where C and D are arbitrary real constants.

4. The x-species is obviously unaffected by the y-species, and grows logistically. The y-species would also grow logistically on its own, but for some reason (perhaps competition for resources, where the x-species always wins) its growth rate is negatively affected by the presence of the x-species.

   Letting $t = c_0\tau$, $x = c_1X$ and $y = c_2Y$ turns the ODEs into

$$\frac{dX}{d\tau} = c_0rX\left(1 - \frac{c_1X}{K}\right), \qquad \frac{dY}{d\tau} = c_0sY\left(1 - \frac{c_2Y}{L}\right) - c_0pc_1XY.$$

   To obtain the given dimensionless form, we must therefore take $c_0 = 1/r$, $c_1 = K$ and $c_2 = L$; the new parameters are then $\alpha = c_0s = s/r$ and $\beta = c_0pc_1 = pK/r$ (and they are obviously also positive). In summary:

$$\tau = rt, \quad X = \frac{x}{K}, \quad Y = \frac{y}{L}, \qquad \alpha = \frac{s}{r}, \quad \beta = \frac{pK}{r}.$$

   A nontrivial equilibrium (X^*, Y^*) occurs where the X-nullcline $X = 1$ intersects the Y-nullcline $\alpha(1-Y) - \beta X = 0$. This gives $(X^*, Y^*) = (1, 1 - \beta/\alpha)$, which lies in the positive quadrant iff $\beta/\alpha < 1$. (In terms of the original parameters this becomes $pK < s$, so the maximal per-capita growth rate s of the y-species must be larger than the negative influence pK that the x-species exerts at its equilibrium population size K.)

5. (a) $\mathbf{x}^*$ is stable if for every neighbourhood U of $\mathbf{x}^*$ there is neighbourhood V of $\mathbf{x}^*$ such that every trajectory starting in V stays in U for all $t \geq 0$.

   (b) $\mathbf{x}^*$ is asymptotically stable if it is stable and has a neighbourhood W such that every trajectory starting in W converges to $\mathbf{x}^*$ as $t \to \infty$.

   (c) $\mathbf{x}^*$ is globally asymptotically stable if it is stable and every trajectory (starting anywhere in $\mathbf{R}^n$) converges to $\mathbf{x}^*$ as $t \to \infty$.

6. With $V(x,y) = 4x^2 + y^2$, which is clearly positive definite, we compute

$$\begin{aligned}\dot V &= 8x \cdot \left(-y + \left(x - \tfrac{3}{4}\right) y^4\right) + 2y \cdot (4x - y^3) \\ &= -2y^4 - 6xy^4 + 8x^2y^4 \\ &= -2y^4(1-x)(1+4x),\end{aligned}$$

which is less than or equal to zero iff $y = 0$ or $-\frac{1}{4} \le x \le 1$, which implies that V is a weak Liapunov function on the open strip

$$\Omega = \left\{(x,y) \in \mathbf{R}^2 : -\tfrac{1}{4} < x < 1\right\}.$$

The set C of points in Ω where $\dot V = 0$ is the line segment $-\frac{1}{4} < x < 1$, $y = 0$, and there we have $\dot y = 4x - 0^3 \neq 0$ except at the equilibrium $(0,0)$. So the non-equilibrium trajectories intersect the line segment C transversally, and can therefore not lie completely in C. The hypotheses for LaSalle's theorem are thus satisfied, so the origin is asymptotically stable.

For any k with $0 < k < 1/4$, the set

$$B = \left\{(x,y) \in \mathbf{R}^2 : V(x,y) \le k\right\} = \left\{(x,y) \in \mathbf{R}^2 : 4x^2 + y^2 \le k\right\}$$

is a topological closed ball (in the shape of an ellipse) contained in Ω, and the minimum of V on the boundary ∂B is clearly k. According to the recipe in LaSalle's theorem, the set

$$N(k) = \left\{(x,y) \in \Omega : V(x,y) < k\right\} = \left\{(x,y) \in \mathbf{R}^2 : 4x^2 + y^2 < k\right\}$$

is therefore a domain of stability, and since this holds for any k in the interval $(0, 1/4)$, we can go all the way up to $k = 1/4$ by taking the union

$$N = \bigcup_{0<k<1/4} N(k) = \left\{(x,y) \in \mathbf{R}^2 : 4x^2 + y^2 < 1/4\right\}.$$

(The argument for this is the usual one: any point in N lies in some $N(k)$, so the trajectory starting there converges to $(0,0)$ and cannot leave $N(k)$; hence it cannot leave N either. So N is indeed a domain of stability.)

For illustration, here is a computer-drawn phase portrait with Ω in gray and N in blue: