Linköpings universitet
Matematiska institutionen
Hans Lundmark

Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2023-08-25 kl. 8.00–13.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Derive the solution formula for the logistic equation $\dot{x} = rx(1 - x/K)$ (where r and K are positive constants) with initial value $x(0) = x_0$.

2. Investigate stability of the equilibrium points and sketch the phase portrait for the system $\dot{x} = y(x-y)$, $\dot{y} = x-2$.

3. Write the system
$$\dot{x} = -y + x(1-x^2-y^2), \qquad \dot{y} = x + y(1-x^2-y^2)$$
in polar coordinates, and use this to sketch the phase portrait.

4. Solve the linear system
$$\dot{x} = -x-2y, \qquad \dot{y} = 2x-5y, \qquad \big(x(0), y(0)\big) = (x_0, y_0)$$
explicitly in the two cases $(x_0, y_0) = (2,2)$ and $(x_0, y_0) = (0,-2)$. Draw the two solution curves $\big(x(t), y(t)\big)$ for $t \ge 0$ in the xy-plane.

5. Consider a bacterial culture growing in a dish, according to the logistic equation $\dot{x} = rx(1-x/K)$, and suppose that a mutation causes another strain of the bacterium to appear, with slightly different properties, so that its growth would be given by $\dot{y} = sy(1-y/L)$ if it were on its own. The situation with both strains growing together may then be modelled by the system
$$\dot{x} = rx\left(1 - \frac{x+y}{K}\right), \qquad \dot{y} = sy\left(1 - \frac{x+y}{L}\right).$$
By studying the phase portrait, determine what conditions the positive parameters (r,s,K,L) must satisfy in order for the mutant strain (y) to outcompete the original strain (x).

6. Compute the general solution of the ODE
$$\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = \frac{1}{1+e^{2t}}.$$

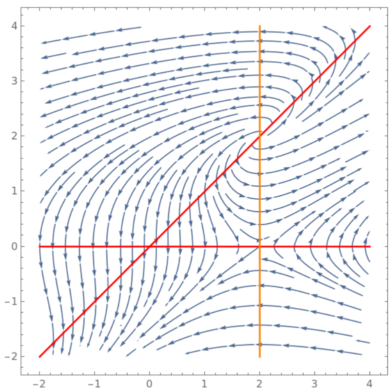
Solutions for TATA71 2023-08-25

1. As we have done in class, use separation of variables, or let $y(t) = 1/x(t)$. The result should be $x(t) = K x_0 \big/ \big(x_0 + (K - x_0) e^{-rt}\big)$.

2. The equilibrium points are $(x, y) = (2, 0)$ and $(2, 2)$. Jacobian matrix:

$$J(x,y) = \begin{pmatrix} y & x-2y \\ 1 & 0 \end{pmatrix} \quad \Longrightarrow \quad J(2,0) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \qquad J(2,2) = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}.$$

By the trace–determinant criterion, $(2, 2)$ is an unstable focus, while $(2, 0)$ is a saddle (which can also be seen from the eigenvalues $\pm\sqrt{2}$, with corresponding principal directions $(\pm\sqrt{2}, 1)$).



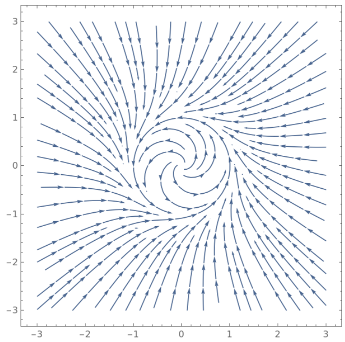
3. With $(x, y) = (r\cos\theta, r\sin\theta)$ we have

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{(x^2+y^2)(1-x^2-y^2)}{r} = r(1-r^2)$$

and

$$\dot{\theta} = \frac{x\dot{y} - x\dot{y}}{r^2} = \frac{x^2+y^2}{r^2} = 1.$$

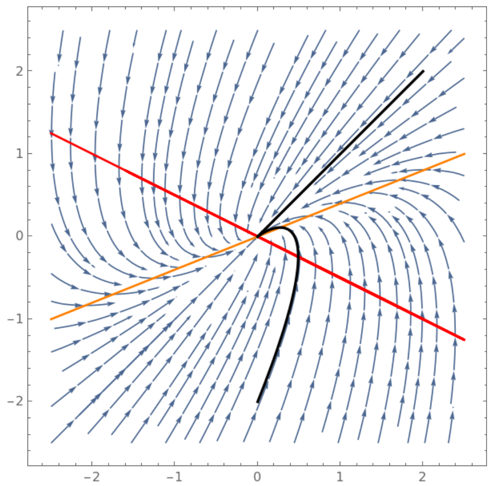
We see that the angle θ increases at a steady rate, while the motion in the radial direction follows the 1-dimensional phase portrait of the ODE $\dot{r} = r(1-r^2)$ for $r \geq 0$: "$0 \longrightarrow 1 \longleftarrow$". So the solution curves are counterclockwise spirals, together with the unit circle (a stable limit cycle) and the equilibrium point at the origin.



(For $r > 1$, the solution curves encircle the origin infinitely many times when followed in the forward direction from a given point on the curve, but not when followed backwards, since in that case $r \to \infty$ in finite time. The solution curves in the region $0 < r < 1$ spiral infinitely many times around the origin both as $t \to +\infty$ and as $t \to -\infty$.)

4. The system's matrix $\left(\begin{smallmatrix}-1 & -2\\ 2 & -5\end{smallmatrix}\right)$ has a double eigenvalue $\lambda = -3$, and it's not a constant multiple of the identity matrix, so the origin is a stable improper node with principal direction given by the corresponding eigenvector $(1,1)$. This immediately implies that $\big(x(t), y(t)\big) = e^{-3t}(2,2)$ is the solution with initial values $(x_0, y_0) = (2,2)$.

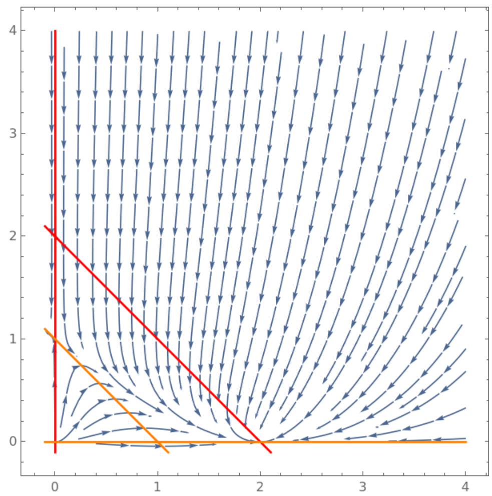
   The change of variables
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & c \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$
   transforms the system into
$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & c \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 & -2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 1 & c \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -3 & 2c \\ 0 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$
   which for $c = 1/2$ takes the canonical form
$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{-3t} \begin{pmatrix} u_0 + t v_0 \\ v_0 \end{pmatrix}.$$
   When $(x_0, y_0) = (0, -2)$ we have $(u_0, v_0) = (-2, 4)$, so the solution in that case is
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-3t} \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 + 4t \\ 4 \end{pmatrix} = e^{-3t} \begin{pmatrix} 4t \\ 4t - 2 \end{pmatrix}$$
   The first curve obviously runs along the line $y = x$. To draw the second curve, it is probably easiest to consider the nullclines and the principal direction, like when sketching the whole phase portrait. Here is a picture with the two curves for $t \geq 0$, drawn in black on top of the phase portrait:



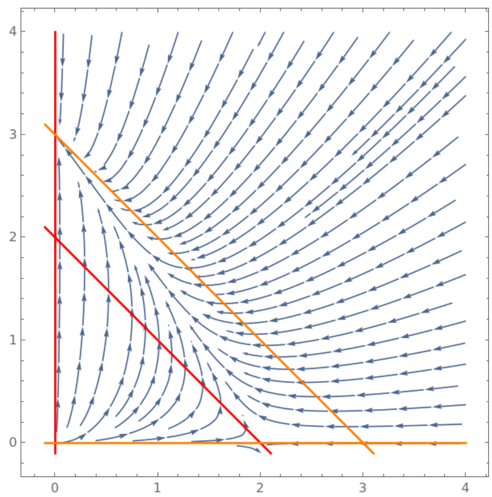
Answer. $\big(x(t), y(t)\big) = e^{-3t}(2,2)$ and $\big(x(t), y(t)\big) = e^{-3t}(4t, 4t-2)$, respectively.

5. For the qualitative outcome, the values of the intrinsic growth rates r and s are immaterial. The only important factor is whether the carrying capacity for the new strain (L) is greater or less than the original one (K), since this determines the relative positions of the nullclines $x + y = K$ and $x + y = L$.

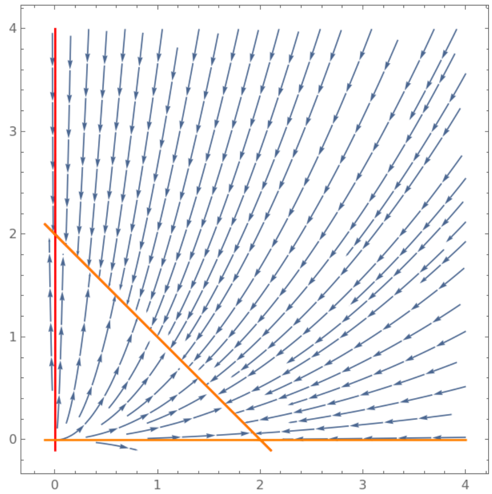
   The figures below are drawn with $(r, s) = (1, 2)$ and $K = 2$. First the case when $L < K$ (illustrated for $L = 1$). In this case, all solutions in the positive quadrant converge towards the equilibrium $(K, 0)$ on the x-axis, so the new strain goes extinct and the old one prevails:

   But if $L > K$ (illustrated with $L = 3$ in the figure), then the new strain outcompetes the old one, since all solutions in the positive quadrant converge to $(0, L)$ instead:

There is also the borderline case $K = L$, which may be ignored, since it "never happens in reality" anyway. But it can be seen that in this case every point on the line $x + y = K$ is an equilibrium, and the initial values (x_0, y_0) and the parameters (r, s) determine which of these equilibria that a given solution curve converges to:



Answer. The new strain outcompetes the old one if and only if $L > K$.

6. From the characteristic polynomial $p(r) = r^2+3r+2 = (r+1)(r+2)$ we find the general solution of the homogeneous equation, $x_{\text{hom}}(t) = Ae^{-t}+Be^{-2t}$, so we seek a solution of the form $x(t) = y_1(t)e^{-t} + y_2(t)e^{-2t}$, using the method of variation of constants. This leads to

$$\begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix} \begin{pmatrix} \dot y_1 \\ \dot y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/(1+e^{2t}) \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} \dot y_1 \\ \dot y_2 \end{pmatrix} = \frac{1}{1+e^{2t}} \begin{pmatrix} e^{t} \\ -e^{2t} \end{pmatrix},$$

so that

$$\begin{aligned} y_1(t) &= \int \frac{e^t}{1+e^{2t}}\,dt = \arctan(e^t) + A, \\ y_2(t) &= \int \frac{-e^{2t}}{1+e^{2t}}\,dt = -\frac{1}{2}\ln(1+e^{2t}) + B. \end{aligned}$$

Anwer. $x(t) = e^{-t}\arctan(e^t) - \frac{1}{2}e^{-2t}\ln(1+e^{2t}) + Ae^{-t} + Be^{-2t}$.