Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2024-01-11 kl. 8.00–13.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3,4,5\}$ you need at least n passed problems and at least $3n-1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Nondimensionalize the following predator–prey system by rescaling the variables:
$$\frac{dx}{dt} = rx - \frac{axy}{m+x}, \qquad \frac{dy}{dt} = -sy + \frac{bxy}{m+x}.$$
Please state clearly how your new dimensionless variables and parameters are related to the original ones.

2. Compute the general solution $\big(x(t), y(t)\big)$ of the linear system
$$\dot{x} = x + 2y, \qquad \dot{y} = x,$$
and draw the phase portrait as carefully as you can.

3. Use linearization to classify the equilibrium points of the system
$$\dot{x} = -2x + y + xy, \qquad \dot{y} = -y + xy,$$
and sketch the phase portrait.

4. Compute the general solution $x(t)$ of the linear ODE
$$(t^2+1)\ddot{x} - (t+1)^2\dot{x} + 2tx = 0.$$
(Hint: $x(t) = e^t$ is a solution.)

5. Show that the origin is an asymptotically stable equilibrium point for the system
$$\dot{x} = -2y^3 + x^3(x^2-4), \qquad \dot{y} = x - y^3,$$
and determine a domain of stability. (Hint: $V(x,y) = x^2 + y^4$.)

6. Let r, K and α be positive constants, and consider the system
$$\dot{x} = rx\left(1 - \frac{x}{K} - \alpha y\right), \qquad \dot{y} = ry.$$
Given a point (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$, compute the flow $\varphi_t(x_0, y_0)$ explicitly for $t \geq 0$. (Hint: Bernoulli equation.)

Solutions for TATA71 2024-01-11

1. After the substitutions $t = c_0\tau$, $x = c_1 X$, $y = c_2 Y$ we get

$$\frac{dX}{d\tau} = c_0 r X - \frac{c_0 c_2 a X Y}{m + c_1 X}, \qquad \frac{dY}{d\tau} = -c_0 s Y + \frac{c_0 c_1 b X Y}{m + c_1 X}.$$

Here various choices are possible. For example, we can take $c_0 = 1/r$, $c_1 = m$ and $c_2 = mr/a$ to obtain

$$\frac{dX}{d\tau} = X - \frac{XY}{1+X}, \qquad \frac{dY}{d\tau} = -\frac{s}{r}Y + \frac{b}{r}\frac{XY}{1+X}.$$

Answer. In terms of the dimensionless variables $\tau = rt$, $X = x/c$, $Y = ay/mr$ the system becomes $dX/d\tau = X - XY/(1+X)$, $dY/d\tau = -\alpha Y + \beta XY/(1+X)$, with the dimensionless parameters $\alpha = s/r$ and $\beta = b/r$.

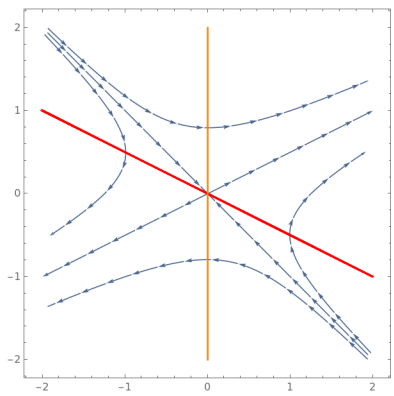
2. Write the system as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix has eigenvalues 2 and -1 with corresponding eigenvectors $\binom{2}{1}$ and $\binom{1}{-1}$, so the origin is a saddle point and the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + B \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t},$$

where A and B are arbitrary constants. Phase portrait:



(The x-nullcline $x + 2y = 0$ and the y-nullcline $x = 0$ are drawn in red and orange, respectively. Note the outgoing and incoming straight line trajectories along the principal directions $\binom{2}{1}$ and $\binom{1}{-1}$. All other solution curves are hyperbolas with these lines as asymptotes.)

Answer. General solution: $x(t) = 2Ae^{2t} + Be^{-t}$, $y(t) = Ae^{2t} - Be^{-t}$. Phase portrait: see above.

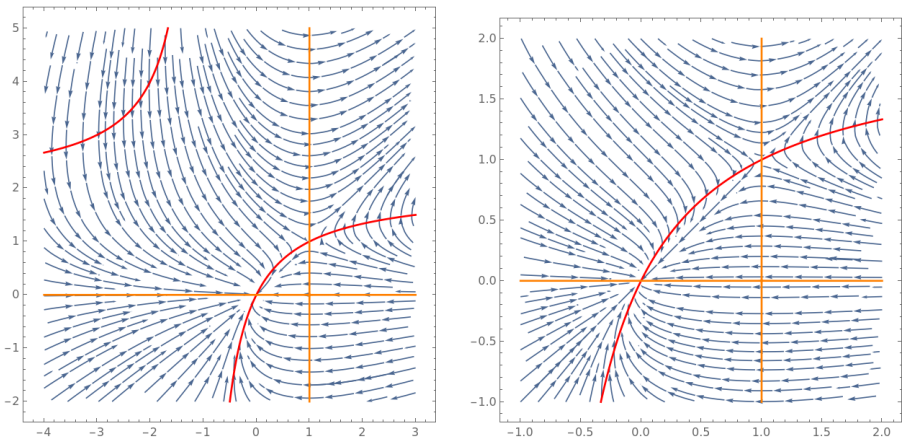
3. The equilibrium points are easily found to be $(0,0)$ and $(1,1)$. Jacobian matrix:

$$J(x,y) = \begin{pmatrix} -2+y & 1+x \\ y & -1+x \end{pmatrix}, \qquad J(0,0) = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}, \qquad J(1,1) = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}.$$

The matrix $J(0,0)$ is triangular, so we read off the eigenvalues -2 and -1 on the diagonal; thus the origin is a stable node. The corresponding eigenvectors are $\binom{1}{0}$ and $\binom{1}{1}$. For $J(1,1)$ we compute the eigenvalues -2 and 1, with the corresponding eigenvectors $\binom{2}{-1}$ and $\binom{1}{1}$, so $(1,1)$ is a saddle point.

The x-nullcline $-2x+y+xy=0$ can be written as $y = \frac{2x}{x+1} = 2 - \frac{2}{x+1}$, a hyperbola with the lines $x=-1$ and $y=2$ as asymptotes. The y-nullcline $0 = -y+xy = y(x-1)$ is the union of the lines $x=1$ and $y=0$.

Taking the signs of $\dot{x}$ and $\dot{y}$ into account, together with the above information about the equilibrium points, we can now sketch the phase portrait (with a zoomed-in version on the right):



(As a bonus piece of information, one may also note that the line $y=x$ is invariant, because of the fact that $\frac{d}{dt}(y-x) = \dot{y} - \dot{x} = -2(y-x)$; indeed, this causes $y-x$ to remain zero if it's zero initially. Moreover, it shows that each solution $(x(t), y(t))$ is such that the perpendicular distance from the point (x,y) to the line $y=x$ decreases as e^{-2t}.)

Answer. There is a stable node at $(0,0)$ (with principal directions $\binom{1}{0}$ and $\binom{1}{1}$) and a saddle point at $(1,1)$ (with principal directions $\binom{2}{-1}$ and $\binom{1}{1}$). Phase portrait: see above.

4. We use reduction of order, setting $x(t) = e^t Y(t)$. This gives $\dot{x} = e^t Y + e^t \dot{Y}$ and $\ddot{x} = e^t Y + 2e^t \dot{Y} + e^t \ddot{Y}$, so that the ODE becomes

$$\begin{aligned} 0 &= (t^2+1)e^t(Y + 2\dot{Y} + \ddot{Y}) - (t+1)^2 e^t (Y + \dot{Y}) + 2te^t Y \\ &= e^t\big((t^2+1)\ddot{Y} + (t^2 - 2t + 1)\dot{Y} + 0Y\big) \\ \iff \quad 0 &= \dot{y} + \left(1 - \frac{2t}{t^2+1}\right) y, \qquad \text{where } y = \dot{Y}. \end{aligned}$$

Here $\exp\big(t - \ln(t^2+1)\big) = e^t/(t^2+1)$ is an integrating factor, so

$$\begin{aligned} \cdots \iff \frac{d}{dt}\left(\frac{e^t}{t^2+1} y\right) = 0 \iff \frac{e^t}{t^2+1} y = -A \\ \iff Y = \int y\,dt = A\int(-t^2-1)e^{-t}dt = A(t^2+2t+3)e^{-t} + B, \end{aligned}$$

which gives us $x(t) = e^t Y(t)$.

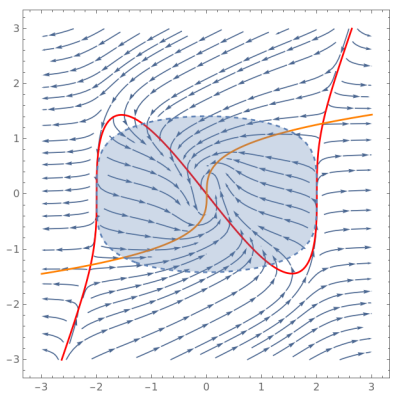
Answer. $x(t) = A(t^2+2t+3) + Be^t$, where A and B are arbitrary constants.

5. The function $V(x,y) = x^2 + y^4$, restricted to the open set

$$\Omega = \{(x,y) \in \mathbf{R}^2 : |x| < 2\},$$

is a strong Liapunov function for this system's equilibrium point $(0,0) \in \Omega$, since $V(0,0) = 0$, $V = x^2 + y^4 > 0$ in $\Omega \setminus \{(0,0)\}$, and $\dot{V} = 2x\dot{x} + 4y^3\dot{y} = 2x(-2y^3 + x^3(x^2-4)) + 4y^3(x - y^3) = -2x^4(4-x^2) - 4y^6 < 0$ in $\Omega \setminus \{(0,0)\}$. Thus Liapunov's theorem shows that $(0,0)$ is asymptotically stable. For $0 < k < 4$, the set $B(k) = \{(x,y) \in \mathbf{R}^2 : x^2 + y^4 \le k\}$ is a closed topological ball inside Ω, and the minimum value of $V(x,y)$ on the boundary $\partial B(k) = \{(x,y) \in \mathbf{R}^2 : x^2 + y^4 = k\}$ is obviously $\alpha = k$ (since that's the *only* value of V on the boundary). Thus, according to the theory, $N(k) = \{(x,y) \in B(k) : V(x,y) < \alpha\} = \{(x,y) \in \mathbf{R}^2 : x^2 + y^4 < k\}$ is a domain of stability whenever $0 < k < 4$, and we can take the union of all these $N(k)$ to get our answer N.

Answer. For proof of stability, see above. $N = \{(x,y) \in \mathbf{R}^2 : x^2 + y^4 < 4\}$ is a domain of stability.



Remark. As the computer-drawn phase portrait shows, the region N (shaded) is not the largest possible domain of stability, but it's what we get with this particular Liapunov function.

6. Computing the flow $\varphi_t(x_0, y_0)$ is the same thing as solving the system of ODEs with the initial value $(x(0), y(0)) = (x_0, y_0)$. From $\dot{y} = ry$ we immediately get $y(t) = y_0 e^{rt}$. Inserting this into the ODE for x gives

$$\dot{x} - r\big(1 - \alpha y_0 e^{rt}\big)x = -\frac{r}{K}x^2,$$

which is a Bernoulli equation ("linear in x on the left-hand side, a power of x on the right-hand side"). The method for solving such equations is to divide by the power of x appearing on the right-hand side. (We don't have to worry about division by zero, since the solution curve starting at (x_0, y_0) with $x_0 > 0$ can't reach the line $x = 0$, where $\dot{x} = 0$.) Dividing by $-x^2$ we get

$$-\frac{\dot{x}}{x^2} + r\big(1 - \alpha y_0 e^{rt}\big)\frac{1}{x} = \frac{r}{K},$$

that is,

$$\dot{z} + r\big(1 - \alpha y_0 e^{rt}\big)z = \frac{r}{K}, \quad \text{where} \quad z(t) = \frac{1}{x(t)}.$$

This is a first-order linear ODE which can be solved using the integrating factor $\exp\big(rt - \alpha y_0 e^{rt}\big)$:

$$\begin{aligned}
\cdots &\iff \frac{d}{dt}\Big(e^{rt}\exp\big(-\alpha y_0 e^{rt}\big)z(t)\Big) = \frac{r}{K}e^{rt}\exp\big(-\alpha y_0 e^{rt}\big) \\
&\iff e^{rt}\exp\big(-\alpha y_0 e^{rt}\big)z(t) = \frac{-1}{K\alpha y_0}\exp\big(-\alpha y_0 e^{rt}\big) + A \\
&\iff z(t) = e^{-rt}\left(\frac{-1}{K\alpha y_0} + A\exp\big(\alpha y_0 e^{rt}\big)\right)
\end{aligned}$$

From $z(0) = 1/x_0$ the constant of integration is determined to be

$$A = \left(\frac{1}{x_0} + \frac{1}{K\alpha y_0}\right)e^{-\alpha y_0},$$

so the solution for x is

$$\begin{aligned}
x(t) = \frac{1}{z(t)} &= \frac{1}{e^{-rt}\left(\dfrac{-1}{K\alpha y_0} + \left(\dfrac{1}{x_0} + \dfrac{1}{K\alpha y_0}\right)e^{-\alpha y_0}\exp\big(\alpha y_0 e^{rt}\big)\right)} \\
&= \frac{e^{rt}}{\dfrac{-1}{K\alpha y_0} + \left(\dfrac{1}{x_0} + \dfrac{1}{K\alpha y_0}\right)\exp\Big(\alpha y_0\big(e^{rt} - 1\big)\Big)}.
\end{aligned}$$

Using the fact that the initial values and the parameters are all positive, it can be verified by inspection that the denominator in this expression is positive for all $t \geq 0$. So the solution does not blow up in finite time; the flow is defined for all $t \geq 0$.

Anwer. $\varphi_t(x_0, y_0) = (x(t), y(t))$ with $x(t)$ as above and $y(t) = y_0 e^{rt}$.