Linköpings universitet
Matematiska institutionen
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Utbildningskod: TATA71
Modul: TEN1

TATA71 Ordinära differentialekvationer och dynamiska system

Tentamen 2024-03-14 kl. 14.00–19.00

No aids allowed, except drawing tools (rulers and such). You may write your answers in English or in Swedish, or some mixture thereof.

Each problem is marked *pass* (3 or 2 points) or *fail* (1 or 0 points). For grade $n \in \{3, 4, 5\}$ you need at least n passed problems and at least $3n - 1$ points.

Solutions will be posted on the course webpage afterwards. Good luck!

1. Nondimensionalize the logistic equation $dx/dt = rx\left(1 - \frac{x}{K}\right)$ (where r and K are positive constants) by a suitable rescaling of the variables x and t. Draw the phase portrait for the nondimensionalized equation.

2. Compute the general solution of the linear system
$$\dot{x} = 3x - 2y, \qquad \dot{y} = -3y,$$
and draw the phase portrait carefully.

3. Use linearization to classify the equilibrium points of the system
$$\dot{x} = x(y - 1), \qquad \dot{y} = y - x^3,$$
and sketch the phase portrait.

4. Show that the origin is an asymptotically stable equilibrium for the system
$$\dot{x} = -y^2 - x^3, \qquad \dot{y} = xy.$$
(Hint: Try a commonly used Liapunov function.) Is the origin *globally* asymptotically stable or not?

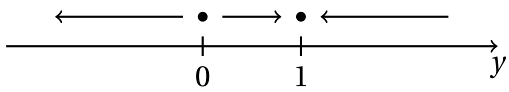
5. Compute the matrix exponential e^{At}, where $A = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix}$ and $t \in \mathbf{R}$.

6. Show that $\Phi(t) = \begin{pmatrix} e^t & 2 \\ 1 & e^{-t} \end{pmatrix}$ is a fundamental matrix for the homogeneous linear system $\dot{\mathbf{x}}(t) = A(t)\,\mathbf{x}(t)$, where $A(t) = \begin{pmatrix} -1 & 2e^t \\ -e^{-t} & 1 \end{pmatrix}$.

   Use this to solve the inhomogeneous system $\dot{\mathbf{x}}(t) = A(t)\,\mathbf{x}(t) + \begin{pmatrix} e^{t/2} \\ 0 \end{pmatrix}$.

Solutions for TATA71 2024-03-14

1. The ODE can be written as $\frac{d(x/K)}{d(rt)} = \frac{x}{K}\left(1 - \frac{x}{K}\right)$, so in terms of the dimensionless variables $y = x/K$ and $s = rt$, it becomes $dy/ds = y(1-y)$. Phase portrait:

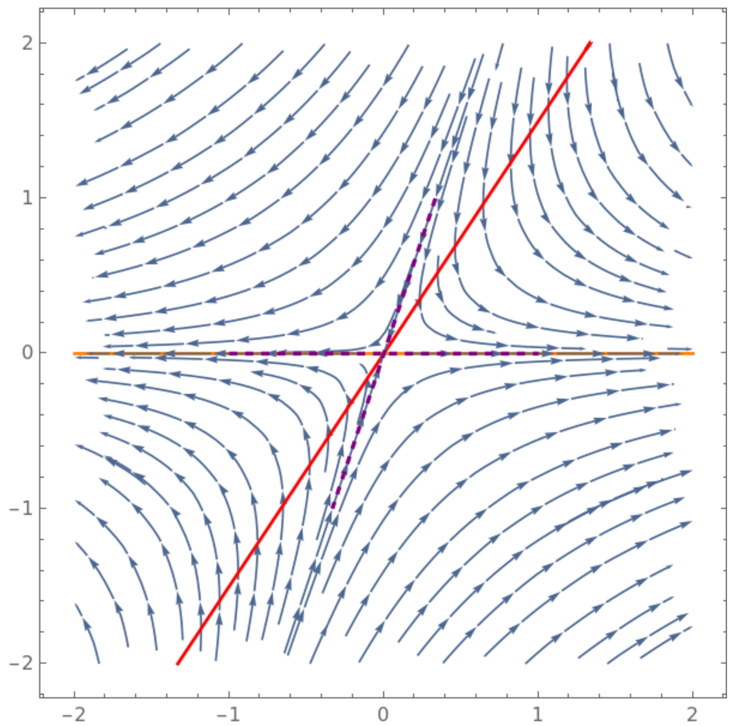


2. The system matrix $A = \begin{pmatrix} 3 & -2 \\ 0 & -3 \end{pmatrix}$ obviously has the eigenvalues 3 and -3, so the phase portrait is a saddle, with principal directions given by the corresponding eigenvectors $\binom{1}{0}$ and $\binom{1}{3}$. From this it follows that the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

where C_1 and C_2 are arbitrary real constants. (Alternative method: First find $y(t)$ from $\dot{y} = -3y$ and then find $x(t)$ from $\dot{x} - 3x = y(t)$.)

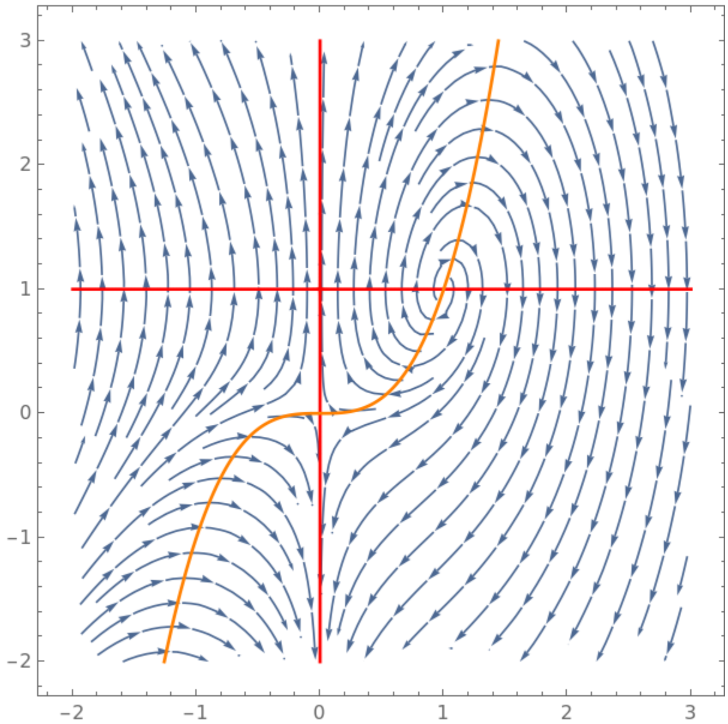
Phase portrait, with the x-nullcline $3x - 2y = 0$ in red and the y-nullcline $y = 0$ in orange, and the principal directions indicated by dashed purple lines:



3. The equilibrium points are easily found to be $(0,0)$ and $(1,1)$. Jacobian matrix:

$$J(x,y) = \begin{pmatrix} y-1 & x \\ -3x^2 & 1 \end{pmatrix}, \qquad J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J(1,1) = \begin{pmatrix} 0 & 1 \\ -3 & 1 \end{pmatrix}.$$

Hence $(0,0)$ is a saddle point with principal directions given by the eigenvectors $\binom{1}{0}$ and $\binom{0}{1}$, while $(1,1)$ is an unstable focus (complex eigenvalues $\frac{1}{2}(1 \pm i\sqrt{11})$ with positive real part). Phase portrait, with x/y-nullclines in red/orange:



4. The function $V(x,y) = x^2 + y^2$ is positive definite and satisfies $\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(-y^2 - x^3) + 2y \cdot xy = -2x^4 \le 0$ for all $(x,y) \in \mathbf{R}^2$, so it's a weak Liapunov function on $\mathbf{R}^2$. The set where $\dot{V} = 0$ is the line $x = 0$, and on this set the ODEs become $(\dot{x}, \dot{y}) = (-y^2, 0)$, so any trajectory passing through a point $(0,y) \neq (0,0)$ on this line is forced to immediately leave the line (towards the left). So the set where $\dot{V} = 0$ contains no complete trajectory except for the equilibrium point itself, and hence the hypotheses for LaSalle's theorem are satisfied, showing that the origin is asymptotically stable. Moreover, since V is a Liapunov function on the whole space $\mathbf{R}^2$ and satisfies the additional condition that $V(x,y) \to \infty$ as $\sqrt{x^2+y^2} \to \infty$, the origin is even globally asymptotically stable.

5. The eigenvalues of A are $-1 \pm 2i = \alpha \pm \beta i$, where $\alpha = -1$ and $\beta = 2$, and an eigenvector corresponding to $\alpha + \beta i = -1 + 2i$ is $\binom{1+i}{1} = \mathbf{a} + \mathbf{b}i$ where $\mathbf{a} = \binom{1}{1}$ and $\mathbf{b} = \binom{1}{0}$. Taking $\mathbf{b}$ and $\mathbf{a}$ as the columns of a change-of-basis matrix M we get the Jordan normal form of A,

$$J = M^{-1}AM = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

so that $A = MJM^{-1}$ and

$$\begin{aligned} \exp(At) &= M \exp(Jt) M^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} e^{\alpha t} \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= e^{-t} \begin{pmatrix} \cos 2t + \sin 2t & -2\sin 2t \\ \sin 2t & \cos 2t - \sin 2t \end{pmatrix}. \end{aligned}$$

6. Calculation shows that $d\Phi/dt = A\Phi$ (both sides are equal to $\operatorname{diag}(e^t, -e^{-t})$), and that $\det(\Phi) = -1 \neq 0$. This means that Φ is a fundamental matrix for the system. We now use the method of variation of constants, setting $\mathbf{x}(t) = \Phi(t)\,\mathbf{y}(t)$. This gives

$$\begin{pmatrix} e^{t/2} \\ 0 \end{pmatrix} = \dot{\mathbf{x}} - A\mathbf{x} = \left(\Phi\dot{\mathbf{y}} + \dot{\Phi}\mathbf{y}\right) - A(\Phi\mathbf{x}) = \Phi\dot{\mathbf{y}} + \underbrace{\left(\dot{\Phi} - A\Phi\right)}_{=0}\mathbf{x} = \Phi\dot{\mathbf{y}},$$

so that

$$\dot{\mathbf{y}} = \Phi^{-1} \begin{pmatrix} e^{t/2} \\ 0 \end{pmatrix} = \begin{pmatrix} -e^{-t} & 2 \\ 1 & e^t \end{pmatrix} \begin{pmatrix} e^{t/2} \\ 0 \end{pmatrix} = \begin{pmatrix} -e^{-t/2} \\ e^{t/2} \end{pmatrix},$$

which after integration becomes

$$\mathbf{y} = \begin{pmatrix} 2e^{-t/2} + C_1 \\ 2e^{t/2} + C_2 \end{pmatrix}.$$

Plugging this back into the defining relation $\mathbf{x} = \Phi\mathbf{y}$, we get the answer:

$$\mathbf{x}(t) = \begin{pmatrix} e^t & 2 \\ 1 & e^{-t} \end{pmatrix} \begin{pmatrix} 2e^{-t/2} + C_1 \\ 2e^{t/2} + C_2 \end{pmatrix} = \begin{pmatrix} 6e^{t/2} \\ 4e^{-t/2} \end{pmatrix} + C_1 \begin{pmatrix} e^t \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ e^{-t} \end{pmatrix}.$$