# Hand-in Exercises TATA74 Curves 1

First of all the exercises are to be solved individually: **it is your examination!**

The exercises to be done by each of you are parametrised by $(M_1 - M_2, D_1 - D_2, Y_1 - Y_2)$, which are the moth, day and year of your birthday. Mine is (1-2, 1-5, 6-3). Some one born March 3 1990 has coordinates (0-3, 0-3, 9-0).

How to get the exercises to be solved by you?: If one Exercise contains exercises of different types, where the types are denoted by letters a, b, c and d parts you must solve one exercise from each of its parts.

When one Exercise contains more than one exercise of a given type (Exercises 4, 5, 6, 7 and 8) you solve the exercise of the type given by the number 1, 2 or 3 obtained as follows:

$$M_1 + M_2 + D_1 + D_2 + Y_1 + Y_2 + \text{ No. of the Exercise } + l \text{ mod} 3$$

where $l = 1$ for an exercise type **a)**, and $l = -1$ for an exercise type **b)**.

So I should solve exercises $a.2$ and $b.3$ in Exercise 4, and $a.3$ and $b.1$ in Exercise 5.

As for Exercise 1, each of you hast to solve 3 of the 10 exercises. In this case you must go to the LISAM-room of the course where all you 11 students are listed (in Members and Groups) where Student no. 1 is the one on top of the list and Student no. 11 is the one at the bottom. With that order you must do:

| Student no. | exercise no. |
|---|---|
| Student no. 1 | 1, 4, 7 |
| Student no. 2 | 1, 8, 9 |
| Student no. 3 | 1, 3, 10 |
| Student no. 4 | 2, 5, 6 |
| Student no. 5 | 6, 4, 10 |
| Student no. 6 | 2, 4, 8 |
| Student no. 7 | 2, 5, 7 |
| Student no. 8 | 3, 10, 7 |
| Student no. 9 | 3, 6, 9 |
| Student no. 10 | 1, 5, 9 |
| Student no. 11 | 5, 8, 10 |

Of course you may use your favourite program to do calculations: MATLAB, Maple, Mathematica, Alpha Wolfram, etc.

## Exercises on Curves

**Exercise 1** *Calculate the curvature and torsion at a generic point of the parametrised curves as well as the length of the following arcs of plane curves:*

1. *Graph of the function* $y = \frac{x^2}{4} - \frac{\ln(x)}{2}$, *with* $x \in [1, 4]$.
2. $\gamma(t) = (8at^3, 3a(2t^2 - t^4))$, *with* $t \in [0, \sqrt{3}]$. $a \neq 0$ *a constant.*
3. *The plane curve* $\gamma(\phi)$ *defined by the polar equation* $r(\phi) = 2a(\cos\phi + 1)$, $\phi \in [0, 2\pi]$. *Remember if* $(r, \phi)$ *are the polar coordinates of a point on the plane, its Cartesian coordinates are* $(x = r\cos(\phi),\ y = r\sin(\phi))$. *This curve is called the cardioid.*
4. *The plane curve* $\gamma(\phi)$ *defined by the polar equation* $r(\phi) = a\sin^3(\phi/3)$ *with* $\phi \in [0, 2\pi]$.
5. *The graph of the function* $y = \ln(\frac{e^x+1}{e^x-1})$ $x \in [t_0, t_1]$.
6. $\gamma(t) = (t, t^2, t^3)$, $t \in [-2, 2]$.
7. $\gamma(t) = (a(t - \sin t), a(1 - \cos t), bt)$, $t \in [-5, 5]$, *with* $a, b$ *non-zero constants.*
8. $\gamma(t) = (a(t - \sin t), a(1 - \cos t), 4a\cos(t/2))$, *with* $t \in [0, 2\pi]$ *and* $a \gneqq 0$ *a constant.*
9. $\gamma(t) = ((\cos t)^3, (\sin t)^3, \cos(2t))$ *for* $t \in (0, \frac{\pi}{2})$.
10. $\gamma(t) = (\sqrt{3}t - \sin t, 2\cos t, t + \sqrt{3}\sin t)$, $t \in [-5, 5]$

**Exercise 2** *Determine the constants* $a \neq 0$ *and* $b \neq 0$ *such that the curvature and the torsion of the curve* $\gamma(t) = (acosh(t), asinh(t), bt)$ *coincide.*

**Exercise 3** **a)** *Determine the point on the curve* $y = 2^x$ *with maximal curvature and calculate the curvature at this point.*

**b)** *Determine the points on the curve* $\gamma(t) = (a(t - \sin t), a(1 - \cos t), 4a\cos(t/2))$ *with local minimal curvature radius and calculate the curvature at these.*

**Exercise 4** *Consider the unit-speed curve* $\gamma(s)$ *with Frenet-trihedron* $\boldsymbol{t}$, $\boldsymbol{n}$ *and* $\boldsymbol{b}$, *curvature* $\kappa \neq 0$ *and torsion* $\tau$. *Show that*

*a.1* $\frac{[\mathbf{n}, \mathbf{n}', \mathbf{n}'']}{|\mathbf{n}'|^2} = \frac{(\frac{\kappa}{\tau})'}{(\frac{\kappa}{\tau})^2 + 1}$

*a.2* $[\mathbf{b}', \mathbf{b}'', \mathbf{b}'''] = \tau^5(\frac{\kappa}{\tau})'$. *Notation:* $(\frac{\kappa}{\tau})'$ *is the derivative of* $(\frac{\kappa}{\tau})$.

*a.3* $[\mathbf{t}', \mathbf{t}'', \mathbf{t}'''] = \kappa^5(\frac{\tau}{\kappa})'$. *Notation:* $(\frac{\tau}{\kappa})'$ *s the derivative of* $\frac{\tau}{\kappa}$.

*b.1 Show that if* $\widehat{\kappa}$ *and* $\widehat{\tau}$ *are the curvature and torsion of the spherical curve* $\widehat{\gamma}(s) = \mathbf{t}(s)$ *then*

$$\widehat{\kappa} = \sqrt{1 + (\frac{\tau}{\kappa})^2} \quad \widehat{\tau} = \frac{(\frac{\tau}{\kappa})'}{\kappa 1 + (\frac{\tau}{\kappa})^2}$$

*b.2* *Let* $\gamma : I \to \mathbb{R}^3$ *be a unit-speed curve with nowhere vanishing torsion* $\tau$. *Consider the curve* $\overline{\gamma} = \int_{s_0}^{s} \mathbf{b}(s)ds$, *called the adjoint curve of* $\gamma$. *(*$\mathbf{b}$ *is the binormal vector to* $\gamma$*). Show that if* $\gamma$ *has constant curvature (resp. torsion), then* $\overline{\gamma}$ *has constant torsion (resp. curvature).*

*b.3* *Let* $\gamma : I \to \mathbb{R}^3$ *be a unit-speed curve with nowhere vanishing constant torsion* $\tau$. *Calculate the curvature of* $\widehat{\gamma}(s) = \frac{-\mathbf{n}}{\tau} + \int_{s_0}^{s} \mathbf{b}(s)ds$.

**Exercise 5** *a.1* *Determine the points on the curve* $\gamma(t) = (3t - t^3, 3t^2, 3t + t^3)$, $t \in \mathbb{R}$, *with tangent line parallel to the plane with equation* $3x + y + z + 2 = 0$.

*a.2* *Find the equation(s) of the tangent line(s) to the curve* $\gamma(t) = (a(t - \sin t), a(1 - \cos t), 4a\sin(\frac{t}{2}))$, *with* $a > 0$ *constant. Which is the angle between a generic tangent line and the axis* $Oz$*?*

*a.3* *Let* $\gamma(\phi)$ *be a regular curve defined by the polar equation* $r = r(\phi)$. *Show that the angle* $\mu$ *formed by the tangent and radial vector to* $\gamma(\phi)$ *is determined by the equation* $\tan\mu = \frac{r}{dr/d\phi}$. *Calculate this angle* $\mu$ *for the cardoid given in Exercise 1.3.*

*b.1* *Show that the curve defined by* $x^2 = 3y$, $2xy = 9z$ *is a helix and determine its axis.*

*b.2* *Show that the curve* $\mathbf{x}(t) = (\cos(t) + 2\sin(t) + 8t/3, 2\cos(t) + \sin(t) - 8t/3, 2\cos(t) + 2\sin(t) - 4t/3)$ *is a helix and determine its axis.*

*b.3* *Show that the curve* $\mathbf{x}(t) = (2t, \ln(t), t^2)$, $t \in \mathbb{R}^+$ *is a helix and determine its axis.*

**Exercise 6** 1. *Determine a plane curve with* $\overline{\kappa} = (1 + s^2)^{-1}$, $s$ *arc-length,*

2. *Determine a plane curve such that* $s = a\tan\phi$, $a$ *constant,* $\phi$ *is the rotation angle of the tangent vector.*

3. *Determine a plane curve such that* $\overline{\kappa} = (1 + s^2)^{-1/2}$, $s$ *arc-length.*

**Exercise 7** 1. *Let* $\mathbf{F} : I \to \mathbb{R}^3$, $I = (-\pi, \pi)$ *be defined by* $\mathbf{F}(t) = (\sin t, \sin t\cos t, \cos^2 t)$. *Determine the unitary tangent vector* $\mathbf{t}$ *to a parametrization* $\gamma : I \to \mathbb{R}^3$ *whose torsion function* $\tau$ *is constant of value 2 and whose binormal vector* $\mathbf{b}(t) = \mathbf{F}(t)$. *(Observe that determines* $\kappa$ *and* $|\tau|$. *Is this true?).*

2. *Integrate the Frenet-Serret equations to show that, if the curvature and the torsion of a regular curve* $\gamma(t)$ *are* $\kappa \neq 0$ *and* $\tau = 1/a$ *(a a constant) , then* $\gamma(t) = a\int g(t) \times g'(t)dt$, *where* $g(t)$ *is a vectorial function satisfying that* $|g(t)| = 1$ *and* $[g, g', g''] \neq 0$.

3. *Using Exercise 4.a.1 show that the normal vector* $\mathbf{n}$ *of a unit-speed curve without inflexion points determine the curvature and torsion of the curve.*

**Exercise 8** *Consider the family of curves defined by* $F(x, y) = a$. *We will consider that* $F_y = \frac{\partial F}{\partial y} \neq 0$

*a.1 Show that* $\nabla F = (F_x, F_y)$ *gives the normal direction to the curve.*

*a.2 Show that the curves which are orthogonal to the curves in the family* $F(x, y) = a$ *is are given by* $\frac{dx}{F_x} = \frac{dy}{F_y}$.

*a.3 Determine the equation of the tangent line and the normal plane (plane parallel to* $Sp\{\mathbf{n}, \mathbf{b}\}$*) to the curve defined by the intecsection of two surfaces:* $F(x, y, z) = 0$, $G(x, y, z) = 0$, *where* $\nabla\Phi = (\Phi_x, \Phi_y, \Phi_z) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(\Phi)$, *the gradient of* $\Phi$.

**b)** *Which curve results of the intersection of* $x^2 + y^2 - z^2 = 1$ *and* $x^2 - y^2 + z^2 = 1$*?*

*b.1 Show that* $\kappa = \frac{|\frac{d^2y}{dx^2}|}{(1+(\frac{dy}{dx})^2)^{3/2}}$.

*b.2 Show that* $\kappa = \frac{|F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2|}{(F_x+F_y)^{3/2}}$.

*b.3 Show that the inflexion points (a point* $\gamma(t_0)$ *is called an inflexion point of the curve if* $\kappa(t_0) = 0$*) of a curve in the family are determined by the equation* $F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2 = 0$, *where* $F_u$ *is the derivative of* $F$ *with respect to the variable* $u$ *and* $F_{uv}$ *is the second derivative of* $F$ *with respect to the variables* $u, v$.

**Exercise 9** *Determine the quadratic Bézier curve* $B(t), 0 \leq t \leq 1$ *joining* $A(M_1.M_2, D_1.D_2)$ *and* $B(1.9, Y_1.Y_2)$ *with horizontal tangent at* $A = B(0)$ *and vertical tangent at* $B = B(1)$. $(M_1 - M_2, D_1 - D_2, Y_1 - Y_2)$ *are the moth, day and year of your birthday. Mine is (1-2, 1-5, 6-3). Some one born March 3 1990 has coordinates (0-3, 0-3, 9-0).*

**Exercise 10** *A cubic Bézier curve* $B(t)$ *is given by the control points* $\mathbf{b}_0 = (0.2, D_1D_2.D_1D_2)$, $\mathbf{b}_1 = (1.0, 0.4)$, $\mathbf{b}_2 = (1.8, 1.2)$ *and* $\mathbf{b}_3 = (3.4, Y_1.Y_2)$.

**a)** *Give the parametric expression of the curve.*

**b)** *Calculate the curvature.*

**c)** *Determine the Bézier curve obtained by reflecting the curve in Exercise 9 in the straight ine* $x + y = 0$.

**Exercise 11** *Use the de Casteljau algorithm on the curve in the previous exercise to calculate* $B(0.M_2M_1)$.

**Exercise 12** **a)** *Consider the curve* $\mathbf{x} = (a\cos t, a\sin t, f(t))$, $f(t)$ *a smooth real function. Determine the condition satisfied by* $f$ *such that the curve* $\mathbf{x}$ *becomes a plane curve.*

**b)** *Is there a plane simple closed curve with length 6 meters and bounding an area of 3 square meters?*

**c)** *Let* $\mathbf{x}(s)$, $s \in [0, l]$, *be a plane simple closed curve such that its curvature satisfies* $0 < \kappa(s) \leq c$, *with* $c$ *a constant. Prove that* $l \geq \frac{2\pi}{c}$.

**d)** *Let* $\mathbf{x}(s)$ *be a plane closed curve with rotation index* $I$ *such that its curvature satisfies* $0 < \kappa(s) \leq c$, *with* $c$ *a constant. Prove that* $l \geq \frac{2\pi I}{c}$, *where* $l$ *is the length of* $\mathbf{x}(s)$.