Hand-in Exercises TATA74 Surfaces 1: Tangent plane, I- and II- fundamental forms.

First of all the exercises are to be solved individually: **it is your examination!**

To every one: no-one is born a 10th, 20th or 30th, in those cases you are born the 11th, 21st or 29th. In the same way nobody is born 1990, 1980 or 2000 but 1989 resp. 1979 or 2005 ..

How to get the exercises to be solved by you?: If one Exercise contains exercises of different types, where the types are denoted by letters a, b, c and d parts you must solve one exercise from each of its parts.

When one Exercise contains more than one exercise of a given type (Exercises 4, 5, 6, 7 and 8) you solve the exercise of the type given by the number 1, 2 or 3 obtained as follows:

$$M_1 + M_2 + D_1 + D_2 + Y_1 + Y_2 + \text{ No. of the Exercise } + l \bmod 3$$

where $l = 1$ for an exercise type **a)**, and $l = -1$ for an exercise type **b)**.

Exercise 1 *Calculate the I-, II-fundamental forms and the Weingarten map of the following surfaces charts:*

a) *(catenoid)* $\mathbf{x} = (y_4 cosh(t/y_4)\cos\theta, y_4 cosh(t/y_4)\sin\theta, y_4 t)$, *$y_4$ the last digit of the year of your birthday,*

b) *(pseudosphere)* $\mathbf{x} = (d_2 \sin t \cos\theta, d_2 \sin t \sin\theta, d_2(lntan(t/2)+\cos t))$, *$d_1 d_2$ the day and y_4 the last digit of the year of your birthday,*

c) *(helicoid)* $\mathbf{x} = (t\cos\theta, t\sin\theta, f(t)+y_4\theta)$, *$y_4$ as above. A helicoid satisfying $f(t) \equiv 0$ is called a right helicoid.*

d) *Hyperbolic paraboloid with equation $z = d_2 xy$, .*

Exercise 2 *a1 Show that all the tangent planes to the surface $z = x^3 + y^3$ at the points $(t, -t, 0)$ contain a straight line*

a2 Write the equations of the tangent plane and the normal to the surface $(u, u^2 - 2uv, u^3 - 3u^2 v)$ at the point (d_1, d_2, y_4), where $d_1 d_2$ the day and y_4 the last digit of the year of your birthday.

a3 Determine the tangent plane to the pseudosphere at a generic point.

b1 Write the equations of the tangent plane and the normal to the surface $\mathbf{x} = (u\cos v, u\sin v, u,$ and the tangent to the curve $u = d_2$ at the point $\mathbf{x}(d_2, \pi/y_4)$ on the same line, where $d_1 d_2$ and y_4 as above.

b2 Consider the surface given by the equation $f(x - az, y - bz) = 0$, with a and b constants. Show that the tangent plane at any point on the surface is parallel to a fixed direction.

b3 A simple surface $\mathbf{r}(s,t)$ is called a surface of translation if $\mathbf{r}(s,t) = \mathbf{x}(s) + \mathbf{y}(t)$. Show that the tangent planes along a parameter curve are parallel to a fixed direction

c Show that the surfaces given by equations $z = tan(x/y)$ and $x^2 - y^2 = a$ are orthogonal.

Exercise 3 *Consider a hyperbolic paraboloid $z = D_2xy$.*

a1 Calculate the angle of intersection of the (straight) parameter lines at a generic point..

a2 Find the equations of the orthogonal curves to the (straight) parameter lines.

a3 Find the equations of the curves that bisect the angle formed by the (straight) parameter lines.

Consider the surface of translation $\mathbf{x}(x,y) = (x + \cos y, x - \cos y, ax)$

b1 Determine the angle between the parameter curves $x = 1$ and $y = \pi/2$.

b2 Show that the tangent line to the curve $x = \sin y$ is also tangent to the curve $x = 1$ at the intersection point.

b3 Show that regions on the paraboloids $z = (x^2 + y^2)/2$ and $z = xy$ that project on regions on the plane with equal area have the same area.

Exercise 4 *Let $\mathbf{x}(x,y) \,|\, x > 0, y > 0$ be a chart to a surface M with coefficients to the first fundamental form $g_{11} = y^2, g_{12} = y, g_{22} = 1 + x^2$. Consider the curves C_1, C_2, C_3 parametrized by $C_i : \alpha_i : (-1,2) \to M$, $\alpha_1(t) = \mathbf{x}(t+1, 2)$, $\alpha_2(t) = \mathbf{x}(1, t+1)$ and $\alpha_3(t) = \mathbf{x}(t+1, t+1)$*

a) *Determine the cosine of the angle between C_1 and C_3 at their intersection point.*

b) *Determine the area of the triangle in M whose sides are on the curves C_1, C_2, C_3.*

Exercise 5 *Consider the surfaces $C = \{(\alpha_1(s), \alpha_2(s), u) \in \mathbb{R}^3 \,|\, s \in \mathbb{R}, u \in \mathbb{R}\}$, $P = \{(x,y,z) \in \mathbb{R}^3 \,|\, y = 0\}$, where $(\alpha_1(s), \alpha_2(s)), s \in \mathbb{R}$ is a unit-speed parametrization of a pane curve. Give a local isometry $f : M \to P$ or show that such an isometry cannot exist.*

Exercise 6 *Let $U = \{(u,v) \in \mathbb{R}^2 \,|\, u > 0\}$ and consider the chart $\mathbf{x}(u,v) = S$ for a surface S. The coefficients of the first fundamental form of S for the chart are $g_{11} = \frac{1+v^2}{u^4}, g_{12} = \frac{-uv}{u^4}, g_{22} = \frac{1}{u^2}$. Consider the map $F : N = \{(x,y,z) \,|\, x > 0, z = 0\} \to S$ given by $F(x,y,z) = \mathbf{x}(\frac{1}{x}, \frac{y}{x})$*

Is F a local isometry?

Exercise 7 **a)** *Consider a surface with equation $z = f(x, y)$. Prove that the surface is a plane if its II-fundamental form is identically 0.*

b) *Determine the II-fundamental form and the asymptotic curves of the tangential surface of an spatial curve $\alpha(s)$: $\mathbf{x}(s,t) = \alpha(s) + t\dot{\alpha}(s)$.*

Exercise 8 *Determine the asymptotic curves on the following surfaces:*

a1 *(catenoid)* $\mathbf{x} = (y_4 cosh(t/y_4)\cos\theta, y_4 cosh(t/y_4)\sin\theta, y_4 t)$, y_4 *the last digit of the year of your birthday,*

a2 $\mathbf{x} = (u - \frac{u^3}{3} + D_2 uv^2, v - \frac{v^3}{3} + D_2 vu^2, y_4(u^2 - v^2))$,

a3 *Compute the principal directions of the surface* $\mathbf{x}(u, v) = (u + D_2 v, y_4 u^2 + y_4 uv, D_2 u)$

b1 *right helicoid,* $\mathbf{x} = (t\cos\theta, t\sin\theta, y_4\theta)$.

b2 *the surface* $\mathbf{x}(u, v) = (u + D_2 v, y_4 u^2 + y_4 uv, D_2 u)$

b3 *Determine the points with Dupin's indicatrix an equilateral hyperbola (asymptotes have equation $y = \pm x$) in the surface* $\mathbf{x}(u, v) = (u + v, u^2 + uv, u)$

Exercise 9 *In applications surfaces are given as level surfaces. One should be able to calculate the characteristic lines of a level surface, then: Let S be a level surface with equation $F(x, y, z) = 0$.*

a) *Show that the asymptotic curves have equations* $dxdF_x + dydF_y + dzdF_z = 0$, $F_x dx + F_y dy + F_z dz = 0$.

b) *Show that the lines of curvature have equations* $F_x dx + F_y dy + F_z dz = 0$,
$$\begin{vmatrix} F_x & dF_x & dx \\ F_y & dF_y & dy \\ F_z & dF_z & dz \end{vmatrix} = 0$$

c) *Show that the geodesics have equations* $F_x dx + F_y dy + F_z dz = 0$, $\begin{vmatrix} F_x & d^2x & dx \\ F_y & d^2y & dy \\ F_z & d^2z & dz \end{vmatrix} = 0$

Exercise 10 **a)** *Show that the set T of points with equation* $\{(x, y, z) \in \mathbb{R}^3 \,|\, z^2 = 1 - (4 - \sqrt{x^2 + y^2})^2\}$ *is a surface. Which surface?*

b) *Show that the curves* $\mathcal{C} = T \cap \{(x, y, z) \in \mathbb{R}^3 \,|\, x = 0, y < 0\}$ *and* $\mathcal{D} = T \cap \{(x, y, z) \in \mathbb{R}^3 \,|\, z = 0\}$ *are lines of principal curvature.*

Exercise 11 *Consider a Bézier surface* $S(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{p} \mathbf{p}_{i,j} B_{i,n}(s) B_{j,p}(t)$, $(s,t) \in [0,1] \times [0,1]$ *of type (n, p).*

a) *Yield the tangent vectors to the parametric curves at the four endpoints of the surface $S(s,t)$ in terms of the control points.*

b) *The normal vector to the Bézier surface $S(s,t)$ at the endpoints are called the endpoint normal vectors of the surface. Give the endpoint normal vectors in terms of the control points.*

Exercise 12 *Consider the Bézier surface $S_{a,b}(s,t)$ of type $(2,2)$ with control points*

$$\begin{array}{lll} \mathbf{p}_{0,0}(0,0,0) & \mathbf{p}_{0,1}(0,1/2,0) & \mathbf{p}_{0,2}(0,1,1/b) \\ \mathbf{p}_{1,0}(1/2,0,0) & \mathbf{p}_{1,1}(1/2,1/2,0) & \mathbf{p}_{1,2}(1/2,1,1/b) \\ \mathbf{p}_{2,0}(1,0,1/a) & \mathbf{p}_{2,1}(1,1/2,1/a) & \mathbf{p}_{2,2}(0,1,1/ab) \end{array}$$

with a, b non-zero constants.

a) *Give the monomial parametrization of the surface. Show that when a and b have the same sign the surface is an elliptic paraboloid, and when a and b have opposite signs the surface is a hyperbolic paraboloid.*

b) *Determine the control points of the first order partial derivatives with respect to s and t of the Bézier surface $S_{a,b}(s,t)$ when $a = Y_2 + M_2$ and $b = Y_2 + D_2$.*