Hand-in Exercises TATA74 Surfaces 2

To every one: no-one is born a 10th, 20th or 30th, in those cases you are born the 11th, 21st or 29th. In the same way nobody is born 1990, 1980 or 2000 but 1989 resp. 1979 or 2005 ..

How to get the exercises to be solved by you?: If one Exercise contains exercises of different types, where the types are denoted by letters a, b, c and d parts you must solve one exercise from each of its parts.

When one Exercise contains more than one exercise of a given type (Exercises 4, 5, 6, 7 and 8) you solve the exercise of the type given by the number 1, 2 or 3 obtained as follows:

$$M_1 + M_2 + D_1 + D_2 + Y_1 + Y_2 + \text{ No. of the Exercise } + l \text{ mod} 3$$

where $l = 1$ for an exercise type **a)**, and $l = -1$ for an exercise type **b)**.

Of course you may use your favourite program to do calculations: MATLAB, Maple, Mathematica, Alpha Wolfram, etc.

Exercise 1 a) *Show that the differential equations for a geodesic* $\gamma(t) = \mathbf{x}(u(t), v(t))$ *that is not parametrised by its arc-length are*

$$\begin{cases} \ddot{u} + \sum \Gamma^1_{ij} \dot{u}\dot{v} &= -\dot{u}\frac{d^2t}{ds^2}(\frac{ds}{dt})^2 \\ \ddot{v} + \sum \Gamma^2_{ij} \dot{u}\dot{v} &= -\dot{v}\frac{d^2t}{ds^2}(\frac{ds}{dt})^2 \end{cases}$$

b) *Let X be a unit vector in $T_P S$. The torsion of a geodesic through a point P with direction X is called* ***the geodesic torsion*** *τ_g in the X-direction. Show that the geodesic torsion to a curve on a chart $\mathbf{x}(u, v)$ can be calculated $\tau_g = [N, d\mathbf{x}, dN]$ (Observe that $\mathbf{x}(u(s), v(s))$)*

c) *Show that a geodesic γ on a chart $\mathbf{x}$ is a line of principal curvature iff γ is a plane curve.*

Exercise 2 *Determine the minimal surfaces given by the equation $z = f(y/x)$, with $x > 0$ and f is a real smooth function.*

Exercise 3 a) *Let P be a point on a surface such that any neighbourhood of P contains two families of geodesics that cut each other under constant angle. Determine K at the point P.*
b)*Show that if (a chart in) a surface contains two families of geodesics that intersect att constant angle then the surface is developable. Determine the Gauss curvature.*

Exercise 4 a) *Show with one example that the mean curvature is not preserved by a local isometry.*

b) *Show that a local isometry that preserves the mean curvature preserves the principal curvatures at corresponding points.*

c) *Show that* $H = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n d\theta$, *where* θ *is the angle given in Euler's Theorem.*

d) *Let* $X, Y \in T_P S$ *be two orthogonal vectors. Show that* $H = \frac{1}{2}(II(X,X) + II(Y,Y))$.

e) *Show that* $H^2 \geq K$. *When does equality hold?*

Exercise 5 *Calculate the Christoffel symbols the geodesics and lines of principal curvature for:*

a1) *A torus* $\mathbf{x}(s,\theta) = ((2+\cos t)\cos\theta, (2+\cos t)\sin\theta, \sin t)$

a2) *A surface with equation* $z = f(x,y)$.

a3) *A surface generate by the binormal lines to a curve* $\alpha(s)$

b1) *catenoid,* $\mathbf{x} = (acosh(t/a)\cos\theta, acosh(t/a)\sin\theta, t)$,

b2) $\mathbf{x}(u,v) = (u+v, u^2+uv, u)$,

b3) *right helicoid,* $\mathbf{x} = (t\cos\theta, t\sin\theta, a\theta)$, *a a constant.*

Exercise 6 *A simple surface is called "of Liouville" if its first fundamental form can be written as* $\begin{pmatrix} f(u)+g(v) & 0 \\ 0 & f(u)+g(v) \end{pmatrix}$.

a) *Show that any surface locally isometric to a surface of revolution is a surface of Liouville.*

b) *Show that the geodesics on this chart can be obtained by integrating* $\int \frac{du}{\sqrt{f-c}} = \pm\int \frac{dv}{\sqrt{g+c}} + d$, *with* c, d *integration constants.*

Exercise 7 **a)** *Let* S *be the embedded surface given as the as the image of the open unit disc in* $\mathbb{R}^2$ *by the chart* $\mathbf{x}(u,v) = (u, v, \ln(1-u^2-v^2))$. *Prove that* $\int_S K dA = 2\pi$.

b) *The habitants of a planet with constant Gauss curvature* K *measurer the angles of a geodesic triangle and get the following values: 34, 62 and 83 degrees. If the area of the triangle is 2,81 units, determine the Gauss curvature. Is the planet spherical?*

Exercise 8 *Suppose we have a Riemannian metric on an open disc* D *of radius* $\delta > 0$ *centred at the origin in* $\mathbb{R}^2$, *possibly with* D *being all in* $\mathbb{R}^2$, *given by the metric tensor in polar coordinates* $\mathbf{x}(r,\lambda)$, $\begin{pmatrix} 1/h^2(r) & 0 \\ 0 & r^2/h(r)^2 \end{pmatrix}$, *where* $h(r) > 0$ *for all* $0 \leq r < \delta$

a) *Prove that*

$$K = hh'' - (h')^2 + \frac{hh'}{r}$$

b) *Write down the geodesic equation for this metric and show that any radial curve, parametrized as to have unit spped, is a geodesic in this metric.*

Exercise 9 *Let* S *be a regular, compact, orientable surface which is not homeomorphic to the sphere. Prove that there are points on* S *where the Gaussian curvature is positive, negative and zero.*

Exercise 10 *Let* $\boldsymbol{x}(u,v)$ *have I fundamental form* $\begin{pmatrix} f(u,v) & 0 \\ 0 & g(u,v) \end{pmatrix}$. *Show that the Gaus curvature is given by* $K = \frac{d\varkappa_1}{ds_1} - \frac{d\varkappa_2}{ds_2} - \varkappa_1^2 - \varkappa_2^2$, *where* $\varkappa_1, \varkappa_2$ *are the geodesic curvatures of the curves with parameter* u *and* v *respectively,* s_1 *is the arc-length of the* u*-curves and* s_2 *is the arc-length of the* v*-curves.*

Exercise 11 *Consider* $S = \{(x,y,z)|x^2+y^2+z^2 = 1, 0 < x, 0 < y, 0 < z\}$ *and the planes ,* $\pi_1 = \{y = 0\}$, $\pi_2 = \{z = 0\}$, $\pi_3 = \{y = x\}$ *and* $\pi_4 = \{\sqrt{2}z = 1\}$. *Let* γ_i *be the intersection of* S *with* π_i, *for* $1 \leq i \leq 4$. *Let* M *be the region enclosed by the union of the arcs* γ_i, T_1 *the spherical triangle bounded by* $\{\pi_1, \pi_2, \pi_3\}$ *and* T_2 *the spherical triangle bounded by* $\{\pi_1, \pi_3, \pi_4\}$. *Calculate the area of* M, T_1 *and* T_2.

Exercise 12 *Decide if there exists a surface with chart* $\mathbf{x}(u,v)$ *and associated functions*

a) $g_{11} = 1$, $g_{22} = 1$, $g_{12} = 0$, $L_{11} = 1$, $L_{22} = -1$ *and* $L_{12} = 0$*?*

b) $g_{11} = 1$, $g_{22} = e^u$, $g_{12} = 0$, $L_{11} = e^u$, $L_{22} = 1$ *and* $L_{12} = 0$*?*