Hand-in Exercises TATA74 1 Fall 2023: Curves

First of all the exercises are to be solved individually: **it is your examination!**

The exercises to be done by each of you are parametrised by $(M_1 - M_2, D_1 - D_2, Y_1 - Y_2)$, which are the moth, day and year of your birthday, but if someone is born year 2000, for this course the student is born 1998. Mine is (1-2, 1-5, 6-3). Some one born March 3 1990 has coordinates (0-3, 0-3, 9-0).

How to get the exercises to be solved by you?: If one Exercise contains exercises of different types, where the types are denoted by letters a, b, c and d parts you must solve one exercise from each of its parts.

When one Exercise contains more than one exercise of a given type (Exercises 1, 2, 5, 6, 7 and 8) you solve the exercise of the type given by the number 1, 2 or 3 obtained as follows:

$$M_1 + M_2 + D_1 + D_2 + Y_1 + Y_2 + \text{ No. of the Exercise } + l \bmod 3$$

where $l = 1$ for an exercise type **a)**, $l = -1$ for an exercise type **b)**, and $l = 0$ for type **c)**.

So I should solve exercises $a.3$ and $b.1$ in Exercise 2, and 1 and 2 in Exercise 6.

Of course you may use your favourite program to do calculations: MATLAB, Maple, Mathematica, Alpha Wolfram, etc.

Exercise 1 *Calculate the curvature and torsion at a generic point of the parametrised curves as well as the length of the following arcs of curves (for plane curves the curvature is signed):*

- *a.1* *Graph of the function* $y = \frac{x}{(1+e^{1/x})}$, *with* $x > 0$.
- *a.2* $\gamma(t) = (a + R\frac{1-t^4}{1+t^2}, b + R\frac{2t}{1+t^2})$, *with* $t \in [0, 1]$. $a, b, R \neq 0$ *constants.*
- *a.3* *The graph of the function* $y = \ln(\frac{1}{\cos(x)})$ $x \in [-\pi/3, \pi/3]$.
- *b.1* *The plane curve* $\gamma(\phi)$ *defined by the polar equation* $r(\phi) = 2a(\cos\phi + 1)$, $\phi \in [0, 2\pi]$. *Remember if* (r, ϕ) *are the polar coordinates of a point on the plane, its Cartesian coordinates are* $(x = r\cos(\phi), y = r\sin(\phi))$. *This curve is called the cardioid.*
- *b.2* *The plane curve* $\gamma(\phi)$ *defined by the polar equation* $r(\phi) = a\sin^3(\phi/3)$ *with* $\phi \in [0, 2\pi]$.
- *b.3* *The plane curve* $\gamma(\phi)$ *defined by the polar equation* $r(\phi) = a\sin^4(\phi/4)$ *with* $\phi \in [0, 2\pi]$.
- *c.1* $\gamma(t) = (e^t\cos(t), e^t\sin(t), 2t)$, $t \in [-\pi, \pi]$.

c.2 $\gamma(t) = (e^t, e^{-t}, \sqrt{2}t)$, $t \in [0,3]$.

c.3 $\gamma(t) = (a(t - \sin t), a(1 - \cos t), 4a\cos(t/2))$, *with* $t \in [0, 2\pi]$ *and* $a \gneq 0$ *a constant.*

d $\gamma(t) = (\sqrt{3}t - \sin t, 2\cos t, t + \sqrt{3}\sin t)$, $t \in [-5, 5]$

Exercise 2 *Consider the unit-speed curve* $\gamma(s)$ *with Frenet-trihedron* $\boldsymbol{t}$, $\boldsymbol{n}$ *and* $\boldsymbol{b}$, *curvature* $\kappa \neq 0$ *and torsion* τ. *Show that*

a.1 $\frac{[\mathbf{n},\mathbf{n}',\mathbf{n}'']}{|\mathbf{n}'|^2} = \frac{(\frac{\kappa}{\tau})'}{(\frac{\kappa}{\tau})^2+1}$

a.2 $[\mathbf{b}', \mathbf{b}'', \mathbf{b}'''] = \tau^5(\frac{\kappa}{\tau})'$. *Notation:* $(\frac{\kappa}{\tau})'$ *is the derivative of* $(\frac{\kappa}{\tau})$.

a.3 $[\mathbf{t}', \mathbf{t}'', \mathbf{t}'''] = \kappa^5(\frac{\tau}{\kappa})'$. *Notation:* $(\frac{\tau}{\kappa})'$ *s the derivative of* $\frac{\tau}{\kappa}$.

b.1 Show that if $\widehat{\kappa}$ *and* $\widehat{\tau}$ *are the curvature and torsion of the spherical curve* $\widehat{\gamma}(s) = \mathbf{t}(s)$ *then*

$$\widehat{\kappa} = \sqrt{1 + (\frac{\tau}{\kappa})^2} \quad \widehat{\tau} = \frac{(\frac{\tau}{\kappa})'}{\kappa(1 + (\frac{\tau}{\kappa})^2)}$$

b.2 Let $\gamma : I \to \mathbb{R}^3$ *be a unit-speed curve with nowhere vanishing torsion* τ. *Consider the curve* $\overline{\gamma} = \int_{s_0}^{s} \mathbf{b}(s)ds$, *called the adjoint curve of* γ. *(*$\mathbf{b}$ *is the binormal vector to* γ*). Show that if* γ *has constant curvature (resp. torsion), then* $\overline{\gamma}$ *has constant torsion (resp. curvature).*

b.3 Let $\gamma : I \to \mathbb{R}^3$ *be a unit-speed curve with nowhere vanishing constant torsion* τ. *Calculate the curvature of* $\widehat{\gamma}(s) = \frac{-\mathbf{n}}{\tau} + \int_{s_0}^{s} \mathbf{b}(s)ds$.

Exercise 3 *Show that the curve* $\gamma(t) = (ae^t\cos(t), ae^t\sin(t), be^t)$ *lies on the cone with equation* $x^2 + y^2 = a^2z^2/b^2$ *with axis the vertical axis. Show that curvature radius at* $\gamma(t)$ *is proportional to the distance from the point* $\gamma(t)$ *to the axis of the cone.*

Exercise 4 *Show that a curve* $\gamma(s)$ *is a helix iff the normal lines to* $\gamma(s)$ *are orthogonal to a fixed direction* $\mathbf{u}$.

Exercise 5 *a.1 Determine the points, with the same parameter* x, *on the curves* $y = x^2$ *and* $y = x^4$ *with parallel tangent lines..*

a.2 Determine the points on $\gamma(t) = (2/t, \ln(t), -t^2)$, $t > 0$, *such that the binormal line to the curve at* $\gamma(t)$ *is parallel to the plane* $x - y + 8z + 2 = 0$

a.3 Let $\gamma(t) = (a(t - \sin t), a(1 - \cos t), 4a\sin(t/2))$, *where* $a > 0$ *a constant. Consider the curve* $\beta(t) = \gamma(t) + a\sqrt{1 + \sin^2(t/2)}\,\mathbf{n}(t)$ *with* $\mathbf{n}$ *the normal vector to* $\gamma(t)$. *Show that this curve is a sinusoide, i.e. the graph of a sine wave.*

b.1 Show that the curve with parametrisation $\gamma(t) = (\sin(2t), 1-\cos(2t), 2\cos(t))$, $-\pi < t < \pi$ *is a spherical curve. Which are the centre and the rdius of the sphere supporting* $\gamma(t)$*?*

b.2 Show that the curve $\gamma(t)$ *with parametrisation* $(16\cos(t)/9 - 32\sin(t)/9 - t/3, 16\cos(t)/9 + 4\sin(t)/9 + 8t/3, 28\cos(t)/9 + 16\sin(t)/9 - 4t/3)$ *is a helix and determine its axis.*

b.3 Let $\gamma(s)$ *be a circular helix. Consider* $\beta(s) = \gamma(s) + \mathbf{b}(s)$*. Show that* $\beta(s)$ *is a helix. Notice that* s *is not the arc-length of* $\beta(s)$

Exercise 6 *a.1 Determine a plane curve with* $\overline{\kappa} = (1+s^2)^{-1}$*,* s *arc-length,*

a.2 Determine a plane curve such that $s = a\tan\phi$*,* a *constant,* ϕ *is the rotation angle of the tangent vector.*

a.3 Determine a plane curve such that $\overline{\kappa} = (1+s^2)^{-1/2}$*,* s *arc-length.*

Exercise 7 *b.1 Let* $\mathbf{F} : I \to \mathbb{R}^3$*,* $I = (-\pi, \pi)$ *be defined by* $\mathbf{F}(t) = (\sin t, \sin t\cos t, \cos^2 t)$*. Determine the unitary tangent vector* $\mathbf{t}$ *to a parametrization* $\gamma : I \to \mathbb{R}^3$ *whose torsion function* τ *is constant of value 2 and whose binormal vector* $\mathbf{b}(t) = \mathbf{F}(t)$*. (Observe that determines* κ *and* $|\tau|$*. Is this true?).*

b.2 Integrate the Frenet-Serret equations to show that, if the curvature and the torsion of a regular curve $\gamma(t)$ *are* $\kappa \neq 0$ *and* $\tau = 1/a$ *(a a constant) , then* $\gamma(t) = a\int g(t) \times g'(t)dt$*, where* $g(t)$ *is a vectorial function satisfying that* $|g(t)| = 1$ *and* $[g, g', g''] \neq 0$*.*

b.3 Using Exercise 2.a.1 show that the normal vector $\mathbf{n}$ *of a unit-speed curve without inflexion points determine the curvature and torsion of the curve.*

Exercise 8 *Consider the family of curves defined by* $F(x,y) = a$*. We will consider that* $F_y = \frac{\partial F}{\partial y} \neq 0$

a.1 Consider the family of curves given by $G(x,y) = b$ *(again* $G_y = \frac{\partial G}{\partial y} \neq 0$*). Show that if the condition* $\frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} = 0$ *is satisfied, then each curve in the first family is orthogonal to each curve of the second family at the intersection point.*

a.2 Give the differential equation for the family of curves formed by those curves that intersect to each curve in the family $F(x,y) = a$ *orthogonally.*

a.3 Deterrmine the family of lines orthogonal to the circles tangent to the x_1*-axis at the origin* O*.*

For the Exercises type b the curves are plane ones.

b.1 Show that $\kappa = \frac{|\frac{d^2y}{dx^2}|}{(1+(\frac{dy}{dx})^2)^{3/2}}$*, for a curve* $y = y(x)$*.*

b.2 *Show that* $\kappa = \frac{|F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2|}{(F_x+F_y)^{3/2}}$. *Give the equation for the inflexion points*

b.3 $\kappa = \frac{r^2+2(\frac{dr}{d\varphi})^2 - r\frac{d^2r}{d\varphi^2}}{(r^2+(\frac{dr}{d\varphi})^2)^{3/2}}$, *for a curve in polar form* $r = r(\varphi)$.

Exercise 9 *Determine the quadratic Bézier curve* $B(t), 0 \leq t \leq 1$ *joining* $A(M_1.M_2, D_1.D_2)$ *and* $B(1.9, Y_1.Y_2)$ *with tangent at* $A = B(0)$ *making a* $\frac{\pi}{4}$-*angle with the* x_1-*axis and horizontal tangent at* $B = B(1)$.

Exercise 10 *A cubic Bézier curve* $B(t)$ *is given by the control points* $\mathbf{b}_0 = (0.2, D_1D_2.D_1D_2)$, $\mathbf{b}_1 = (1.0, 0.4)$, $\mathbf{b}_2 = (1.8, 1.2)$ *and* $\mathbf{b}_3 = (3.4, Y_1.Y_2)$.

a) *Give the parametric expression of the curve.*

b) *Calculate the curvature.*

c) *Determine the Bézier curve obtained by rotating the curve an angle* $\frac{\pi}{4}$ *with centre the origin in* $\mathbb{R}^2$.

Exercise 11 *Use the de Casteljau algorithm on the curve* $B(t)$ *in the previous exercise to subdivide the curve in two Bézier curves that meet at the point with parameter* $B(0.5Y_2)$.

Exercise 12 **a)** *Show that the equations for the envelope of the* ***uniparametric*** *family of plane curves given by* $F(x, y, a, b) = 0$, *where the parameters* a, b *satisfy the condition* $\varphi(a, b) = 0$ *are*

$$F(x, y, a, b) = 0, \quad \varphi(a, b) = 0, \quad det(\frac{\partial(F, \varphi)}{\partial(a, b)}) = 0$$

($\frac{\partial(F,\varphi)}{\partial(a,b)}$ *is the Jacobian of the function* $(F(\cdot, \cdot, a, b), \varphi(a, b))$)

b) *We know that the envelope of the family of straight lines* $ax + y + b = 0$ *is the circle with equation* $x^2 + y^2 = c^2$, *with* a, b *parameters and* c *a constant. Give the condition satisfied by* a *and* b *(the function* $\varphi(a, b) = 0$*).*

c) *Let* $\mathbf{x}(s)$, $s \in [0, l]$, *be a plane simple closed curve such that its curvature satisfies* $0 < \kappa(s) \leq c$, *with* c *a constant. Prove that* $l \geq \frac{2\pi}{c}$.

d) *Let* $\mathbf{x}(s)$ *be a plane closed curve with rotation index* I *such that its curvature satisfies* $0 < \kappa(s) \leq c$, *with* c *a constant. Prove that* $l \geq \frac{2\pi I}{c}$, *where* l *is the length of* $\mathbf{x}(s)$.