# Hand-in Exercises TATA74 Surfaces 2

To every one: no-one is born a 10th, 20th or 30th, in those cases you are born the 11th, 21st or 29th. In the same way nobody is born 1990, 1980 or 2000 but 1989 resp. 1979 or 2005 ..

How to get the exercises to be solved by you?: If one Exercise contains exercises of different types, where the types are denoted by letters a, b, c and d parts you must solve one exercise from each of its parts.

When one Exercise contains more than one exercise of a given type (Exercises 4, 5, 6, 7 and 8) you solve the exercise of the type given by the number 1, 2 or 3 obtained as follows:

$$M_1 + M_2 + D_1 + D_2 + Y_1 + Y_2 + \text{ No. of the Exercise } + l \bmod 3$$

where $l = 1$ for an exercise type **a)**, $l = -1$ for an exercise type **b)**, and $l = 0$ for an exercise of type **c)**.

Of course you may use your favourite program to do calculations: MATLAB, Maple, Mathematica, Alpha Wolfram, etc.

**Exercise 1 a)** *Show that the differential equations for a geodesic $\gamma(t) = \mathbf{x}(u(t), v(t))$ that is not parametrised by its arc-length are*

$$\left\{ \begin{array}{rcl} \ddot{u} + \sum \Gamma^1_{ij} \dot{u}\dot{v} & = & -\dot{u} \frac{d^2t}{ds^2} (\frac{ds}{dt})^2 \\ \ddot{v} + \sum \Gamma^2_{ij} \dot{u}\dot{v} & = & -\dot{v} \frac{d^2t}{ds^2} (\frac{ds}{dt})^2 \end{array} \right\}.$$

**b)** *Let $X$ be a unit vector in $T_P S$. The torsion of a geodesic through a point $P$ with direction $X$ is called **the geodesic torsion** $\tau_g$ in the $X$-direction. Show that the geodesic torsion to a curve on a chart $\mathbf{x}(u, v)$ can be calculated $\tau_g = [N, d\mathbf{x}, dN]$ (Observe that $\mathbf{x}(u(s), v(s))$)*

**c)** *Show that a geodesic $\gamma$ on a chart $\mathbf{x}$ is a line of principal curvature iff $\gamma$ is a plane curve.*

**Exercise 2** *Determine the minimal surfaces given by the equation $z = f(y/x)$, with $x > 0$ and $f$ is a real smooth function.*

**Exercise 3 a1)** *Let $P$ be a point on a surface such that any neighbourhood of $P$ contains two families of geodesics that cut each other under constant angle. Determine $K$ at the point $P$.*

**a2)** *Show that if (a chart in) a surface contains two families of geodesics that intersect att constant angle then the surface is developable. Determine the Gauss curvature.*
*We say that a surface $\mathbf{x}$ is developable if $\mathbf{x}(s, t) = \gamma(t) + s\mathbf{a}(t)$, with $\mathbf{a}$ a unit vector, and for every $t$, $\mathbf{x}$ satisfies $[\gamma', \mathbf{a}, \mathbf{a}'] = 0$.*

**a3)** *Consider a chart $\mathbf{x}$ aaround a point $P(0, 0, 0)$ in a surface $S$ such that $\gamma_1$ and $\gamma_2$ are two curves tangent to $S$ at $P = \gamma_1(0) = \gamma_2(0)$. The osculating* *plane for both curves at $P$ is the plane $\{(x,y,z) \in \mathbb{R}^3; z=0\}$. Assume that the curvature of $\gamma_1$ at $P$ is 1, and the curvature of $\gamma_2$ at $P$ is 2. what can we say on the tangent plane to $S$ at $P$?*

**Exercise 4** **a1)** *Let $S$ be a surface such that the curve $\gamma(t)$ with support $\{(x(t), y(t), z(t)); x=y=z^2\}$ is a geodesic. Determine the tangent plane to $S$ at $P(1,1,1)$.*

**a2)** *Consider the chart $\mathbf{x} : \mathcal{U} = \mathbb{R} \times (-\pi/2, \pi/2) \to \mathbb{R}^3$given by $\mathbf{x}(t,\theta) = ((t^2+1)\cos(\theta), (t^2+1)\sin(\theta), t^3+t)$. Is any value of $a \in \mathbb{R}$ such that the curve $\gamma(\theta) = ((a^2+1)\cos(\theta), (a^2+1)\sin(\theta), a^3+a)$ is a geodesic in $\mathbf{x}(\mathcal{U})$?*

**a3)** *Consider the chart $\mathbf{x}(t,\theta) = ((\cos(t)+2)\cos(\theta), (\cos(t)+2)\sin(\theta), t)$. Determine the geodesics that are plane curves contained in the plane $\{z=0\}$.*

**b1)** *Consider the chart $\mathbf{x}(t,\theta) = ((\cos(t)+2)\cos(\theta), (\cos(t)+2)\sin(\theta), t)$. Show that the plane curves $\mathbf{x}(\mathcal{U}) \cap \{z=0\}$ are lines of principal curvature.*

**b2)** *Consider the chart $\mathbf{x}(t,\theta) = ((\cos(t)+2)\cos(\theta), (\cos(t)+2)\sin(\theta), t)$. Show that the plane curves $\mathbf{x}(\mathcal{U}) \cap \{ax+by=0\}$, with $a, b$ constants, are lines of principal curvature.*

**b3)** *Consider the torus $T$ with equation $z^2 = 1-(4-\sqrt{x^2+y^2})^2$. Show that the plane curves $\mathcal{C}_1 = T \cap \{z=0\}$ and $\mathcal{C}_2 = T \cap \{(x,y,z) \in \mathbb{R}^3; x=0, y>0\}$ are lines of principal curvature.*

**Exercise 5** *Calculate the Christoffel symbols the geodesics and lines of principal curvature for:*

**a1)** *A torus $\mathbf{x}(s,\theta) = ((2+\cos t)\cos\theta, (2+\cos t)\sin\theta, \sin t)$*

**a2)** *A surface with equation $z = f(x,y)$.*

**a3)** *A surface generate by the binormal lines to a curve $\alpha(s)$*

**b1)** *catenoid, $\mathbf{x} = (acosh(t/a)\cos\theta, acosh(t/a)\sin\theta, t)$,*

**b2)** $\mathbf{x}(u,v) = (u+v, u^2+uv, u)$,

**b3)** *right helicoid, $\mathbf{x} = (t\cos\theta, t\sin\theta, a\theta)$, $a$ a constant.*

**c1)** $\mathbf{x} = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$

**c2)** *Let $\boldsymbol{x}(u,v)$ be a parametrised surface with I fundamental form given by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & \cos^2 u \end{pmatrix}$. Calculate the Christoffel symbols, the Gaussian curvature and the equation of the geodesics.*

**c3)** *Show that if the plane curve $\gamma(s) = \mathbf{x}(u(s), v(s))$ is a geodesic, not a straight line, then $\gamma(s) = \mathbf{x}(u(s), v(s))$ is a line of principal curvature in $\boldsymbol{x}(u,v)$.*

**Exercise 6** *Consider a chart $\mathbf{x}(u,v)$ where the lines of principal curvature are the parametric curves. Show that the geodesic curvature of the parametric curves are: $\kappa_g(dv=0) = \frac{1}{\sqrt{g_{11}}}\frac{d\mathbf{t}_1}{du} \cdot \mathbf{t}_2 = -\frac{1}{\sqrt{g_{11}}}\frac{d\mathbf{t}_2}{du} \cdot \mathbf{t}_1$, $\kappa_g(du=0) = \frac{1}{\sqrt{g_{22}}}\frac{d\mathbf{t}_2}{dv} \cdot \mathbf{t}_1 = -\frac{1}{\sqrt{g_{22}}}\frac{d\mathbf{t}_1}{dv} \cdot \mathbf{t}_2$.*

**Exercise 7** **a)** *Let $S$ be the graph of a smooth function $F$ (i.e. $S$ has a parametrization $z = F(x,y)$) Determine the Gaussian curvature of $S$ in terms of the derivatives of $F$.*

**b)** *Consider the hyperboloid of two sheets with equation $x^2 + y^2 = z^2 - 1$. Show that the Gaussian curvature is everywhere positive.*

**c)** *Consider the surface $z = e^{-(x^2+y^2)/2}$. Calculate the Gaussian curvature at a generic point.*

**d)** *Determine the points in the surface of part c) where the Gaussian curvature is positive.*

**Exercise 8** **a)** *Let $S$ be the embedded surface given as the as the image of the open unit disc in $\mathbb{R}^2$ by the chart $\mathbf{x}(u,v) = (u, v, \ln(1 - u^2 - v^2))$. Prove that $\int_S K dA = 2\pi$.*

**b)** *The habitants of a planet with constant Gauss curvature $K$ measurer the angles of a geodesic triangle and get the following values: 34, 62 and 83 degrees. If the area of the triangle is 2,81 units, determine the Gauss curvature. Is the planet spherical?*

**Exercise 9** *Consider the surface $\mathbf{x}(u,v)$ in geodesic coordinates and I fundamental form $\begin{pmatrix} 1 & 0 \\ 0 & g(u,v) \end{pmatrix}$. Show that if $\gamma(s) = \mathbf{x}(u(s), v(s))$ is a geodesic, then $0 = \frac{d\theta}{ds} + \frac{\partial\sqrt{g}}{\partial u}\frac{dv}{ds}$, where $\boldsymbol{g}_1 = \frac{\mathbf{x}_u}{|\mathbf{x}_u|}, \boldsymbol{g}_2 = \frac{\mathbf{x}_v}{|\mathbf{x}_v|}$ are the unitary tangent vectors to the parametric curves and $\gamma'(s) = \mathbf{g}_1 \cos\theta + \mathbf{g}_2 \sin\theta$, where $\theta$ is the angle between $\gamma'(s)$ and $\boldsymbol{g}_1$.*

**Exercise 10** *Consider the surface $\boldsymbol{x}(u,v)$ with I fundamental form $\begin{pmatrix} f(u) & 0 \\ 0 & g(u) \end{pmatrix}$. Show that the curves $v = \pm\int \frac{a\sqrt{f(u)}}{\sqrt{g(u)}\sqrt{g(u)-a^2}} du$, with $a = const.$ are geodesics.*

**Exercise 11** *Consider $S = \{(x,y,z) | x^2 + y^2 + z^2 = 1, 0 < x, 0 < y, 0 < z\}$ and the planes , $\pi_1 = \{y = 0\}$, $\pi_2 = \{z = 0\}$, $\pi_3 = \{y = x\}$ and $\pi_4 = \{\sqrt{2}z = 1\}$. Let $\gamma_i$ be the intersection of $S$ with $\pi_i$, for $1 \le i \le 4$. Let $M$ be the region enclosed by the union of the arcs $\gamma_i$, $T_1$ the spherical triangle bounded by $\{\pi_1, \pi_2, \pi_3\}$ and $T_2$ the spherical triangle bounded by $\{\pi_1, \pi_3, \pi_4\}$. Calculate the area of $M$, $T_1$ and $T_2$.*

**Exercise 12** *Decide if there exists a surface with chart $\mathbf{x}(u,v)$ and associated functions*

**a1)** $g_{11} = 1$, $g_{22} = 1$, $g_{12} = 0$, $L_{11} = 1$, $L_{22} = -1$ *and* $L_{12} = 0$*?*

**a2)** $g_{11} = 1$, $g_{22} = e^u$, $g_{12} = 0$, $L_{11} = e^u$, $L_{22} = 1$ *and* $L_{12} = 0$*?*

**a3)** *Calculate the Riemann tensor for a plane, a circular cylinder and a sphere.*

*Let $R^h_{ijk} = L_{ik}L^h_j - L_{ij}L^h_k$ be the coefficients to the Riemann tensor. Show that*

**b)** $R^h_{ijk} = -R^h_{ikj}$ *och så* $R^h_{ijj} = 0$, *för alla $i$, $j$, $k$ och $h$.*

**c)** $R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0$, *för alla $i$, $j$, $k$ och $h$.*