Bézier Curves (1968-1977)


Pierre Bèzier

Bézier's PhD Thesis (in maths) on the **study of parametric polynomial curves and their vector coefficients**. Bézier presented his PhD thesis in 1977, after his years at Renault.

With that he could translate mathematical and computing tools into computer aided design -he is the father of UNISURF CAD/CAM- and three dimensional modeling.

Beginning of the history: Weierstrass proved (1885) his Approximation Theorem: Any continuous real valued function defined on an interval can be uniformly approximated as closely as desired by a polynomial function.

Berstein (1912) gave another proof with elementary methods, and explicitly gave the approximation polynomials using "his" polynomials as a basis for the vector space of polynomials up to degree n. Thus, the **polynomial p is a linear combinations of Berstein polynomials, with real coeffcients**.

Idea: If we change the real coefficient by vector coefficients we get: **the uniformly approximation (as closely as want) of a curve by polynomial curves, the Bézier curves**.
We can do even better: **we can approximate with polygons (the control polygons), which are piece-linear maps**. And now we can render the curve.

If we consider now a family of Bézier curves that swipes a (patch of a) surface, Bézier surface, we can easily render sobjects in 3D. You will see ruled surface in the Spring.

Bézier Curves. Properties of Bernstein Polynomials

A Bézier curve is a curve $\mathbb{B}(t) : [0,1] \longrightarrow \mathbb{R}^3$ with <u>control points</u> $\mathbb{b}_0, \dots, \mathbb{b}_n$ is a polynomial curve defined by

$$\mathbb{B}(t) = \sum_{i=0}^{n} \mathbb{b}_i \, B_{i,n}(t) \qquad t \in [0,1]$$

where $B_{i,n}(t) = \begin{cases} \binom{n}{i}(1-t)^{n-i} \cdot t^i & \text{if } 0 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$

The polynomials $B_{i,n}(t)$ are called Bernstein polynomials

The polygon formed by $\bigcup_{i=1}^{n} [\mathbb{b}_{i-1}, \mathbb{b}_i]$ is called the <u>control</u> polygon

Example i) $B_{0,1}(t) = (1-t)$, $B_{1,1}(t) = t$ Curve $\equiv$ Control polygon

ii) $B_{0,2}(t) = (1-t)^2$, $B_{1,2}(t) = 2(1-t)t$, $B_{2,2}(t) = t^2$

iii) $B_{0,3}(t) = (1-t)^3$, $B_{1,3}(t) = 3(1-t)^2 t$, $B_{2,3}(t) = 3(1-t)t^2$, $B_{3,3}(t) = t^3$

If we recall some properties of binomial coefficients:

i) $\binom{n}{i} = \binom{n}{n-i}$

ii) $\binom{n+1}{i+1} = \binom{n}{i} + \binom{n}{i+1}$

iii) For any natural number n and any pair of real numbers x, y:
$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

We get some very useful properties of Bernstein's polynomials:

i) $\forall\, t \in [0,1]$ $B_{i,n}(t) \geq 0$ for $0 \leq i \leq n$

$\binom{n}{i} > 0$, $(1-t)^{n-i} \geq 0$ and $t^i \geq 0$

Given $n \geq 1$

ii) $\sum_{i=0}^{n} B_{i,n}(t) = 1$ since

$$\sum_{i=0}^{n} B_{i,n}(t) = \sum_{i=0}^{n} \binom{n}{i}(1-t)^{n-i} t^i = ((1-t)+t)^n = 1.$$

iii) $B_{n-i,n}(t) = \binom{n}{n-i}(1-t)^i t^{n-i} = \binom{n}{i}[1-(1-t)]^{n-i}(1-t)^i$

$= B_{i,n}(1-t)$

iv) Recursion formula: $B_{i,n}(t) = (1-t)B_{i,n-1}(t) + t B_{i-1,n-1}(t)$
for $i = 0, \dots, n$, where $B_{-1,n-1}(t) \equiv 0 \equiv B_{n,n-1}(t)$

$$\begin{cases} B_{i,n-1}(t) = \binom{n-1}{i}(1-t)^{n-1-i} t^i \\ B_{i-1,n-1}(t) = \binom{n-1}{i-1}(1-t)^{n-i} t^{i-1} \end{cases}$$

For $i=0$ $B_{0,n}(t) = (1-t)^n = (1-t)(1-t)^{n-1} = (1-t)B_{0,n-1}(t)$
$= (1-t)B_{0,n-1}(t) + B_{-1,n-1}(t)\, t$

For $i=n$ $B_{n,n}(t) = t^n = t\,(t)^{n-1} = t\,B_{n-1,n-1}(t) =$
$= t\,B_{n-1,n-1}(t) + (1-t)B_{n,n-1}(t)$

For $1 \leq i \leq n-1$

$(1-t)B_{i,n-1}(t) + t B_{i-1,n-1}(t) = (1-t)^{n-i}\binom{n-1}{i}t^i + t^i\binom{n-1}{i-1}(1-t)^{n-i}$

$= (1-t)^{n-i} t^i \left(\binom{n-1}{i} + \binom{n-1}{i-1}\right) = \binom{n}{i}(1-t)^{n-i} t^i = B_{i,n}(t)$

v) $\sum_{i=0}^{n} \frac{i}{n} B_{i,n}(t) = t$

$\sum_{i=0}^{n} \frac{i}{n} B_{i,n}(t) = \sum_{i=1}^{n} \frac{i}{n} B_{i,n}(t) = \sum_{i=1}^{n-1} \frac{i}{n} B_{i,n}(t) + B_{n,n}(t)$

$= \sum_{i=1}^{n-1} \frac{i}{n}\binom{n}{i}(1-t)^{n-i} t^i + t B_{n-1,n-1}(t) =$

$= t \sum_{i=1}^{n-1} \binom{n-1}{i-1}(1-t)^{n-1-(i-1)} t^{i-1} = t\left[((1-t)+t)^{n-1}\right] = t$

vi) Derivatives of Bernstein polynomials

To calculate the curvature of a Bézier curve we need to calculate the first and second derivative of the Bernstein polynomials

a) $$\frac{dB_{i,n}(t)}{dt} = \frac{(i-nt)}{t(1-t)} B_{i,n}(t)$$

a') $$\frac{dB_{i,n}(t)}{dt} = n\left(B_{i-1,n-1}(t) - B_{i,n-1}(t)\right)$$

b) $$\frac{d^2 B_{i,n}(t)}{dt^2} = \left(\frac{i(i-1) - 2i(n-1)t + n(n-1)t^2}{t^2(1-t)^2}\right) B_{i,n}(t)$$

Proof a) Differentiating $B_{i,n}(t) = \binom{n}{i}(1-t)^{n-i} t^i$

a) $$\frac{dB_{i,n}(t)}{dt} = \binom{n}{i}\left[-(n-i)(1-t)^{n-i-1} t^i + i(1-t)^{n-i} t^{i-1}\right]$$

a) $$= \binom{n}{i}(1-t)^{n-i} t^i \left[\frac{(i-n)}{(1-t)} + \frac{i}{t}\right] = \binom{n}{i}(1-t)^{n-i} t^i \left(\frac{i-nt}{t(1-t)}\right)$$

a'): $$\frac{dB_{i,n}(t)}{dt} = -n\binom{n-1}{i}(1-t)^{n-1-i} t^i + n\binom{n-1}{i-1}(1-t)^{n-i} t^{i-1}$$

$$= -n B_{i,n-1}(t) + n B_{i-1,n-1}(t)$$

b) $$\frac{d^2 B_{i,n}(t)}{dt^2} = \frac{d}{dt}\left(\left(\frac{i-nt}{t(1-t)}\right) B_{i,n}t\right)$$

Applying formulas a') and b) we have that if $\mathbf{B}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_{i,n-1}(t)$ is a Bézier curve

then

1) $\dot{\mathbb{B}}(t) = \sum_{i=0}^{n-1} \mathbb{b}_i^{(1)} B_{i,n-1}(t)$ with $\mathbb{b}_i^{(1)} = n(\mathbb{b}_{i+1} - \mathbb{b}_i)$

2) $\ddot{\mathbb{B}}(t) = \sum_{i=0}^{n-2} \mathbb{b}_i^{(2)} B_{i,n-2}(t)$ with

$\mathbb{b}_i^{(2)} = n(n-1)(\mathbb{b}_{i+2} - 2\mathbb{b}_{i+1} + \mathbb{b}_i) = (n-1)(\mathbb{b}_{i+1}^{(1)} - \mathbb{b}_i^{(1)})$

We prove 1)

$$\dot{\mathbb{B}}(t) = \frac{d}{dt}\left(\sum_{i=0}^{n} \mathbb{b}_i B_{i,n}(t)\right) = \sum_{i=0}^{n} \mathbb{b}_i n\left(B_{i-1,n-1}(t) - B_{i,n-1}(t)\right)$$

$$= \sum_{j=0}^{n-1} B_{j,n-1}(t)\left(n\,\mathbb{b}_{j+1} - n\,\mathbb{b}_j\right).$$

Remember $B_{n,n-1}(t) = 0$.

To prove 2) just apply 1) to $\dot{\mathbb{B}}(t) = \sum_{i=0}^{n-1} \mathbb{b}_i^{(1)} B_{i,n-1}(t)$

In general we have

$$\mathbb{B}^{(r)}(t) = \sum_{i=0}^{n-r} \mathbb{b}_i^{(r)} B_{i,n-r}(t) \quad \text{with}$$

$$\mathbb{b}_i^{(r)} = n(n-1)\cdots(n-r+1)\left(\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} \mathbb{b}_{i+j}\right)$$

Properties of Bézier Curves

The properties of Bernstein polynomials used to define a Bézier curve $\mathbb{B}(t) : [0,1] \longrightarrow \mathbb{R}^3$

$\mathbb{B}(t) = \sum_{i=0}^{n} \mathbb{b}_i B_{i,n}(t)$ with control pts $\mathbb{b}_0, \dots, \mathbb{b}_n$

are passed to the curve so:

Endpoints Interpolation $\mathbb{B}(0) = \mathbb{b}_0 = \sum_{i=0}^{n} \mathbb{b}_i \binom{n}{i}(1-t)^{n-i} t^i$

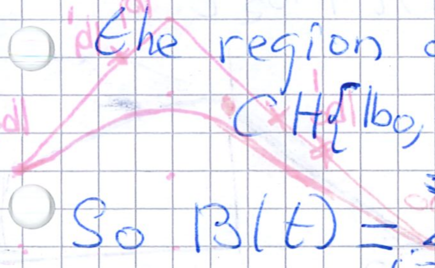
$= \mathbb{b}_0 \binom{n}{0} (1)^n = \mathbb{b}_0$

$\mathbb{B}(1) = \sum_{i=0}^{n} \mathbb{b}_i (1-t)^{n-i} t^i = \mathbb{b}_n \binom{n}{n} (1)^n = \mathbb{b}_n$

Endpoint Tangent $\dot{\mathbb{B}}'\big|_0 = n(\mathbb{b}_1 - \mathbb{b}_0)$

$\dot{\mathbb{B}}'\big|_1 = n(\mathbb{b}_n - \mathbb{b}_{n-1})$ applying the endpoint interpolation to the Bézier curve which is $\dot{\mathbb{B}}(t)$

Convex Hull Property Given points $\mathbb{b}_0, \dots, \mathbb{b}_n$ in $\mathbb{R}^3$. The convex hull CH of $\{\mathbb{b}_0, \dots, \mathbb{b}_n\}$ is the region of $\mathbb{R}^3$ given by



$$CH\{\mathbb{b}_0, \dots, \mathbb{b}_n\} = \left\{ x \in \mathbb{R}^3 \; ; \; x = \mathbb{b}_0 \lambda_0 + \dots + \mathbb{b}_n \lambda_n \; ; \; \begin{matrix} \lambda_i \geq 0 \\ \sum \lambda_i = 1 \end{matrix} \right\}$$

So $\mathbb{B}(t) = \sum_{i=0}^{n} \mathbb{b}_i B_{i,n}(t) \in CH\{\mathbb{b}_0, \dots, \mathbb{b}_n\}$ since

$\lambda_i(t) = B_{i,n}(t) \geq 0$, $t \in [0,1]$ and $\sum_{i=0}^{n} B_{i,n}(t) = 1$

Affine Invariance Let $\mathbb{B}(t)$ be a ~~3D~~ (2D) Bézier curve with control points $\mathbb{b}_0, \dots, \mathbb{b}_n$ then an affine transformation f of ~~3D~~ (resp. 2D) maps the curve to another Bézier curve with control points $\{f\mathbb{b}_0, \dots, f\mathbb{b}_n\}$. Observe that $\mathbb{B}(t)$ is a linear combination of $\{\mathbb{b}_0, \dots, \mathbb{b}_n\}$.

For planar Bézier Curves **Variation Diminishing Property**

Let $\mathbb{B}(t) = \sum_{i=0}^{n} b_i B_{i,n}(t)$, $b_i \in \mathbb{R}^2$ be a planar Bézier curve. Then the number of intersections of a given line with $\mathbb{B}(t)$ is less than ~~or equal to~~ the number of intersections of that line with the controll polygon

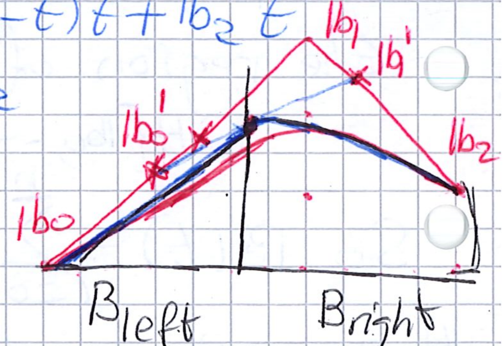
The Casteljeau algorithm gives us a method of subdivision of Bezier curves

Th (Subdivision) Given a Bézier curve $\mathbb{B}(t) = \sum_{i=0}^{n} b_i B_{i,n}(t)$ with control pts $b_0, \dots, b_n$, the control points of the two curve arcs obtained by subdivision at parameter value t are $\underset{B'(t)}{b_0^0}, b_0^1, b_0^{n-1}, \underset{B''(t)}{b_0^n}$ for $B_{left}(t)$ $\left(B'(t)\right)$ and $b_0^n, b_1^{n-1}, b_2^{n-2}, \dots, b_{n-1}^1, b_n^0$ for Bright $B''(t)$

Example Let $\mathbb{B}(t) = b_0(1-t)^2 + 2b_1(1-t)t + b_2 t^2$

When evaluating $\mathbb{B}(0.35)$ we have

$b_0^0(1.0, 1.0)$, $b_1^0(8, 6)$, $b_2^0(12, 2)$



$b_0^1 = (1-0.35)b_0^0 + 0.35\, b_1^0 = (3.45, 2.75)$

$b_1^1 = (1-0.35)b_1^0 + 0.35\, b_2^0 = (9.4, 4.6)$

$b_0^2 = (1-0.35)b_0^1 + 0.35\, b_1^1 = \mathbb{B}(0.35) = (5.53, 3.4)$

Control points for B_{left} are $b_0(1,1)$, $b_0^1(3.45, 2.75)$ and $b_0^2 = \mathbb{B}(0.35) = (5.53, 3.4)$

Control points for B_{right} are $b_0^2(5.53, 3.4) = \mathbb{B}(0.35)$, $b_1^1(9.4, 4.6)$ and $b_2^0(12, 2)$

Conversion of Bézier curves to monomial curves

Bézier curves are polynomial curves, if instead of using Bernstein polynomials as a basis for $\mathbb{P}_n$ = Space of polynomials of degree $\leq n$

we use the basis $\{1, t, \dots, t^n\}$ we write the curve with parametrization

$$\alpha(t) = a_0 + a_1 t + \dots + a_n t^n$$

$a_0, \dots, a_n$ are called the monomial control pts

For instance a cubic Bézier curve has expression

$$\mathbb{B}(t) = \mathbb{b}_0 \overbrace{(1-t)^3}^{(1-3t+3t^2-t^3)} + \mathbb{b}_1 3(t - 2t^2 + t^3) + \mathbb{b}_2 3(t^2 - t^3) + \mathbb{b}_3 t^3$$

$$= (\mathbb{b}_0) + (3\mathbb{b}_1 - 3\mathbb{b}_0)t + (3\mathbb{b}_0 - 6\mathbb{b}_1 + 3\mathbb{b}_2)t^2 + (-\mathbb{b}_0 + 3\mathbb{b}_1 - 3\mathbb{b}_2 + \mathbb{b}_3)t^3$$

So
$$\begin{cases} a_0 = \mathbb{b}_0 \\ a_1 = 3\mathbb{b}_1 - 3\mathbb{b}_0 \\ a_2 = 3\mathbb{b}_0 - 6\mathbb{b}_1 + 3\mathbb{b}_2 \\ a_3 = -\mathbb{b}_0 + 3\mathbb{b}_1 - 3\mathbb{b}_2 + \mathbb{b}_3 \end{cases}$$ or

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & +3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{b}_0 \\ \mathbb{b}_1 \\ \mathbb{b}_2 \\ \mathbb{b}_3 \end{pmatrix} = T_{con} \begin{pmatrix} \mathbb{b}_0 \\ \mathbb{b}_1 \\ \mathbb{b}_2 \\ \vdots \\ \mathbb{b}_n \end{pmatrix}$$

Observe that it comes directly from the change of basis in the space of polynomials

$$\begin{pmatrix} B_{0,3} \\ B_{1,3} \\ B_{2,3} \\ B_{3,3} \end{pmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} = T^t_{con} \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^3 \end{pmatrix}$$

of course $\begin{pmatrix} 1 \\ t \\ \vdots \\ t^3 \end{pmatrix} = T^{t^{-1}}_{con} \begin{pmatrix} B_{0,3} \\ \vdots \\ B_{3,3} \end{pmatrix}$

and $\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = T_{con}^{-1} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 3 & 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$

Example. The control points of the cubic Bézier curve $B(t)$ given in monomial form

$\alpha(t) = (1+t^2+t^3,\ 2+3t-t^3,\ 6t-3t^2)$ with monomial control points $a_0(1,2,0)$ $a_1(0,3,6)$, $a_2(1,0,-3)$ and $a_3(1,-1,0)$

The control points of the Bézier curve are

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 1 & 0 & -3 \\ 1 & 2 & 0 \end{pmatrix} = \begin{cases} b_0(1,2,0) \\ b_1(1,3,2) \\ b_2(4/3,4,3) \\ b_3(3,7,3) \end{cases}$$

$$\boxed{B(t) = (1,2,0)(1-t)^3 + (1,3,2)\,3(1-t)^2 t + (4/3,4,3)\,3(1-t)t^2 + (3,7,3)\,t^3}$$

In general notice that $B_{i,n}(t) = \sum_{j=0}^{n-i} \binom{n}{i} (-1)^{n-i-j} \binom{n-i}{j} t$

So T_{con} is the lower triangular matrix with entries $T_{con}(i,j) = \begin{cases} \binom{n}{i}\binom{i}{j}(-1)^{i-j} & i \geq j \\ 0 & \text{otherwise} \end{cases}$

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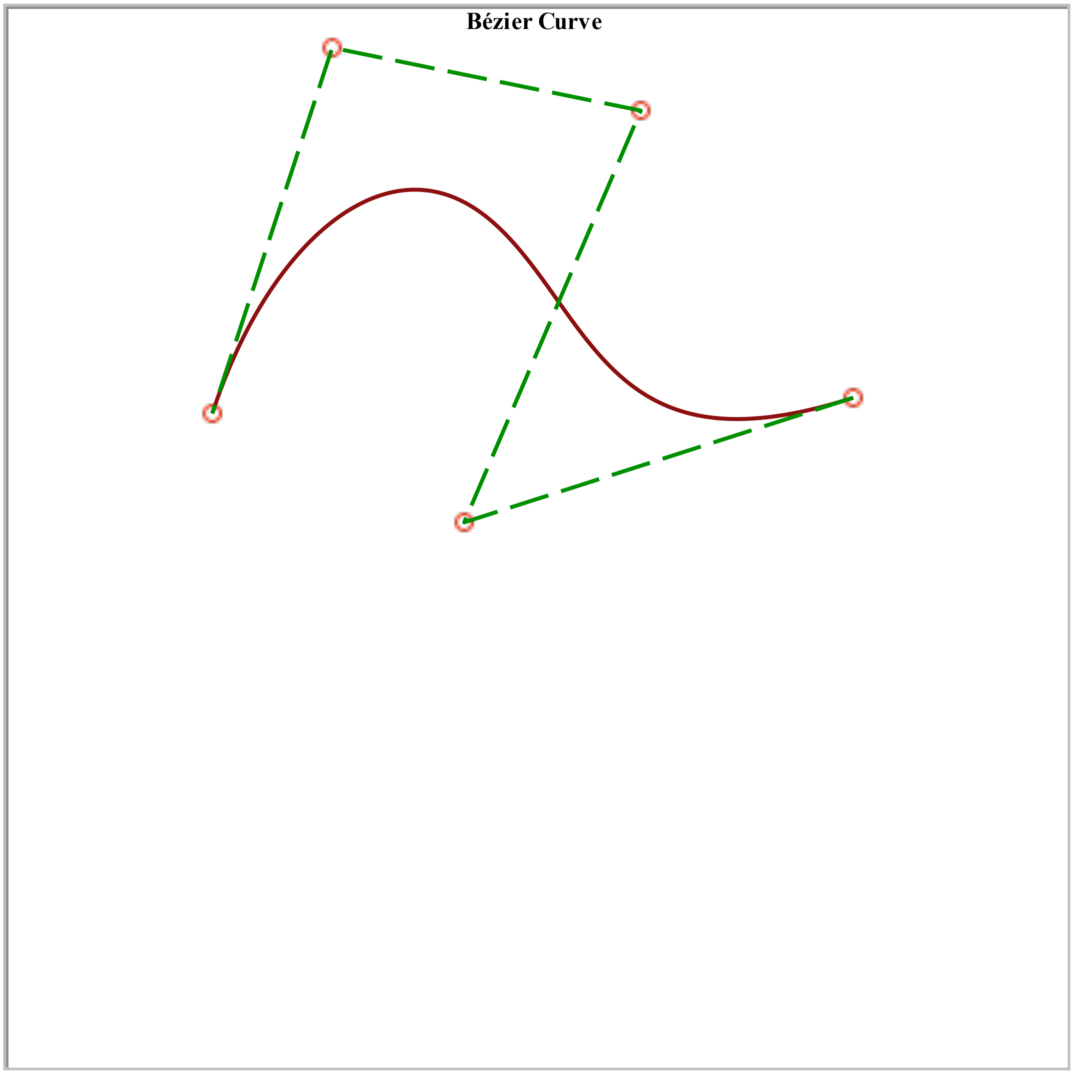
and T_{con}^{-1} is the lower triangular matrix with entries $T_{con}^{-1}(i,j) = \begin{cases} \binom{i}{j} / \binom{n}{j} & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$

So to transform the quartic Bézier curve with control pts $b_0(0,0,0)$, $b_1(1,2,5)$, $b_2(2,-1,-1)$, $b_3(1,0,0)$, $b_4(0,0,1)$ we get the monomial control pts

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 6 & -12 & 6 & 0 \\ -4 & 12 & -12 & 0 \\ 1 & 4 & 6 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 5 \\ 2 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



☑ Control points	☑ Control curve	☐ Convex Hull	Reset

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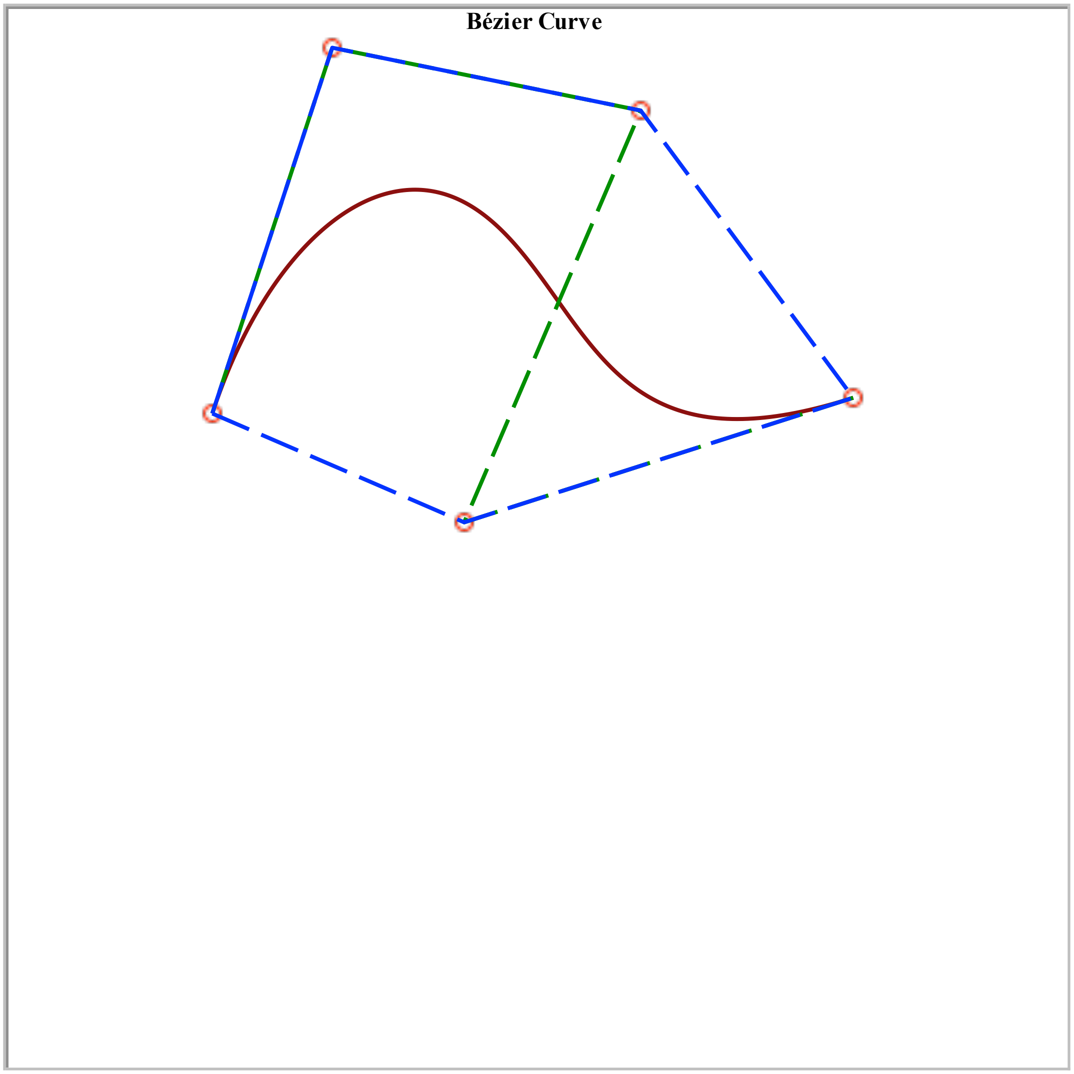

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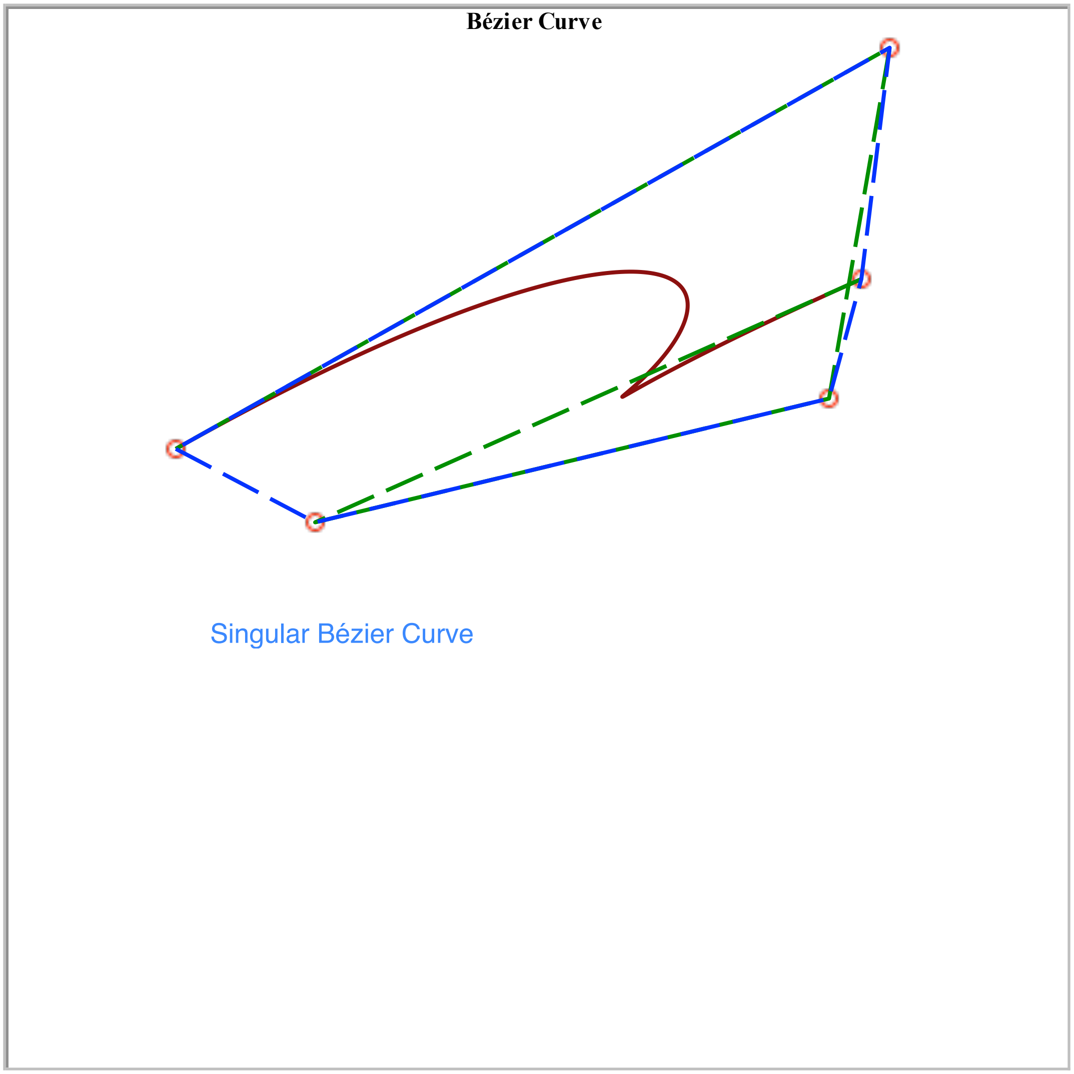

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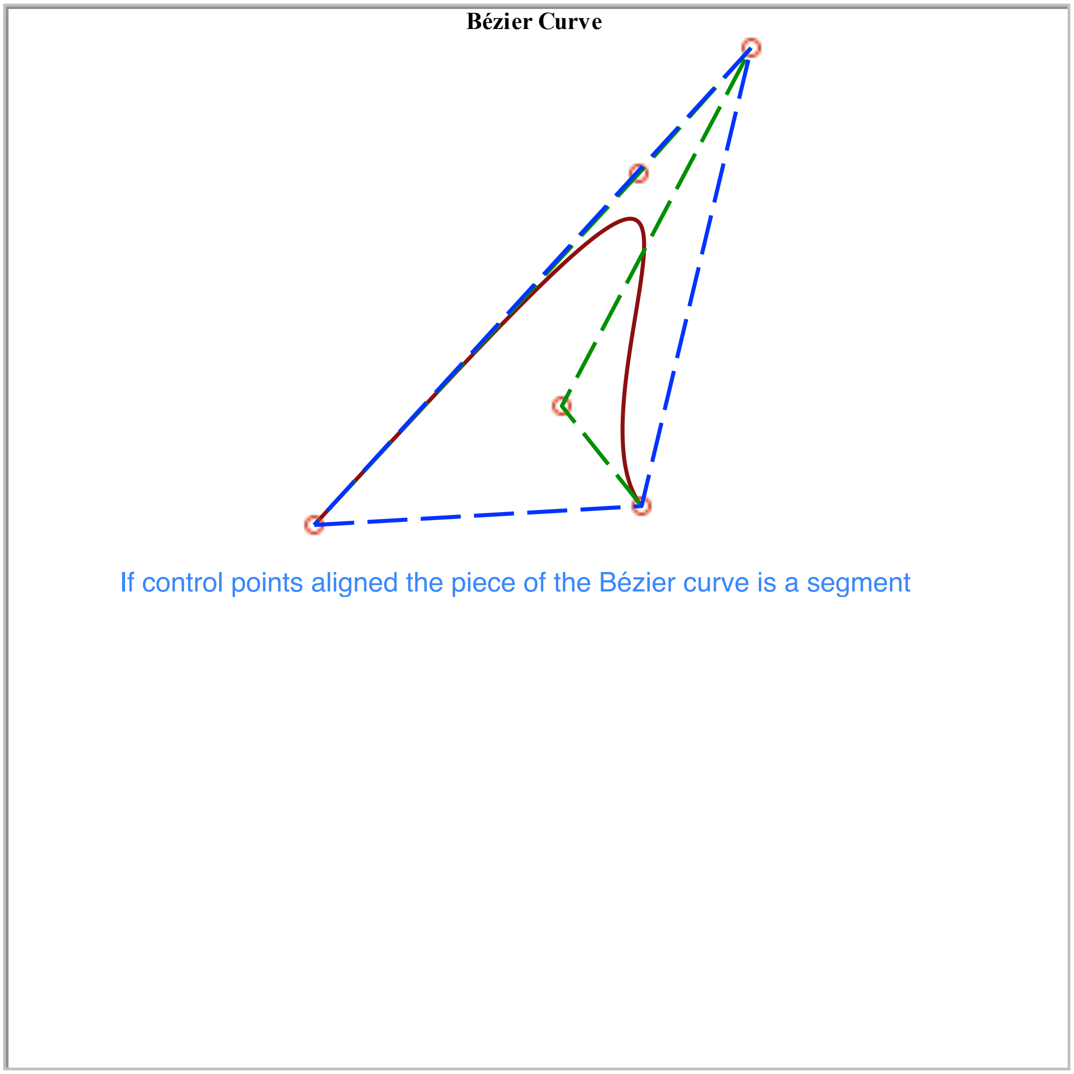

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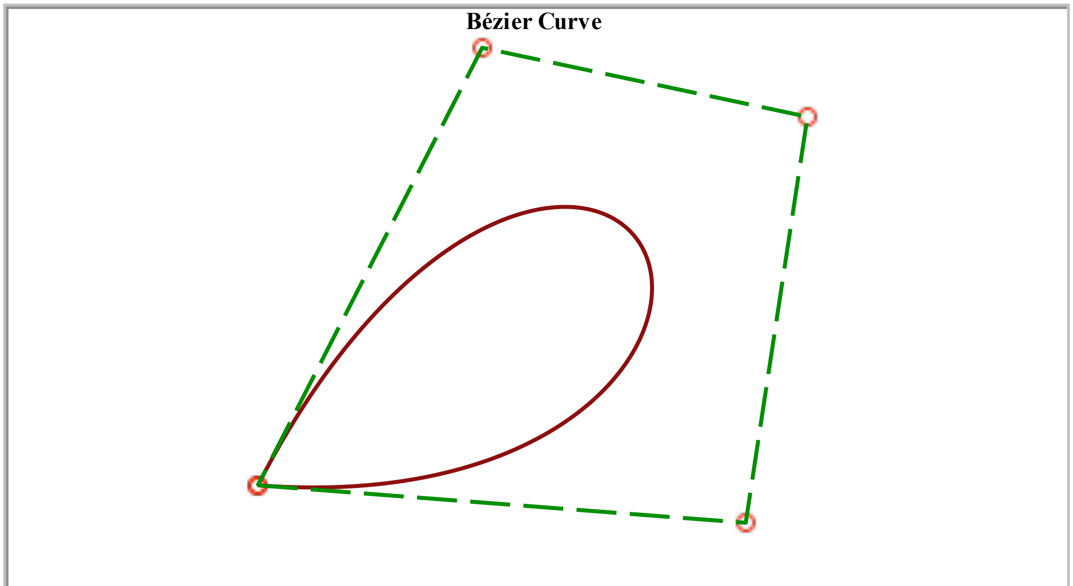

Closed Bézier Curve

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Example Consider the cubic Bézier curve with control points $P_0(1,0)$, $P_1(2,3)$, $P_2(5,4)$, $P_3(2,1)$
And consider a rotation around the origin anticlockwise through an angle $\pi/4$. The resulting Bezier curve is the Bezier curve with control points $Q_0(0.707, 0.707)$, $Q_1(-0.707, 3.536)$
$Q_2(0.707, 6.364)$ and $Q_3(0.707, 2.121)$

Since we have

$$(q_0, q_1, q_2, q_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 & 2 \\ 0 & 3 & 4 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The de Casteljau Algorithm

Method of evaluating the point on a Bézier curve corresponding to the parameter value $t \in [0,1]$
Very useful to plot (rendering) the curve

Example
Consider a ~~quadra~~ quadratic Bézier curve with controll points $b_0(1.0, 1.0)$, $b_1(8.0, 6.0)$ and $b_2(12.0, 2.0)$

de Casteljau Algorithm gives, for $t = 0.35$

$b_0^0 = b_0 = (1, 1)$, $b_1^0 = b_1 = (8, 6)$ and $b_2^0 = b_2 = (12, 2)$

$b_0^1 = (1-0.35)b_0^0 + 0.35 b_1^0 = (0.65 + 2.80, 0.65 + 2.10) = (3.45, 2.75)$

$b_1^1 = (1-0.35)b_1^0 + 0.35 b_2^0 = (5.2 + 4.2, 3.9 + 0.7) = (9.4, 4.6)$

And finally $B(0.35) = 0.65 b_0^1 + 0.35 b_1^1 = \cancel{(6.11+1.61,}$

$= (2.24 + 3.29, 1.79 + 1.61)$

$= (5.53, 3.4)$