# Bézier Surfaces

Recall – Bézier curve of degree $n$ is given by $B(s) = \sum_{i=0}^{n} \mathbb{P}_i B_{i,n}(s)$, with $\mathbb{P}_0, \dots, \mathbb{P}_n$ control points and $B_{i,n}(s) = \binom{n}{i}(1-s)^{n-i} s^i$
$0 \leq i \leq n$

Bézier surface (case of graph-surface) is given by $S(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{p} \mathbb{P}_{i,j} B_{i,n}(s) B_{j,p}(s)$
$(s,t) \in [0,1] \times [0,1]$

$\mathbb{P}_{0,0}, \dots, \mathbb{P}_{n,p}$ the control points
$(\{\mathbb{P}_{0,0}, \dots, \mathbb{P}_{n,p}\}$ : control polyhedron$)$

Observe that a Bézier surface is the "product" of Bézier curves

Example : Bilinear Bézier surface with $\mathbb{P}_{0,0}(1,2,3)$, $\mathbb{P}_{0,1}(4,2,4)$, $\mathbb{P}_{1,0}(2,3,1)$, $\mathbb{P}_{1,1}(6,4,2)$

$$B(s,t) = \begin{pmatrix}1\\2\\3\end{pmatrix}(1-t)(1-s) + \begin{pmatrix}4\\2\\4\end{pmatrix}(1-s)t + \begin{pmatrix}2\\3\\1\end{pmatrix}s(1-t) + \begin{pmatrix}6\\4\\2\end{pmatrix}ts$$

$$= \left[\begin{pmatrix}1\\2\\3\end{pmatrix}(1-s) + \begin{pmatrix}2\\3\\1\end{pmatrix}s\right](1-t) + \left[\begin{pmatrix}4\\2\\4\end{pmatrix}(1-s) + \begin{pmatrix}6\\4\\2\end{pmatrix}s\right]t$$

$$= \left[\begin{pmatrix}1\\2\\3\end{pmatrix}(1-t) + \begin{pmatrix}4\\2\\4\end{pmatrix}t\right](1-s) + \left[\begin{pmatrix}2\\3\\1\end{pmatrix}(1-t) + \begin{pmatrix}6\\4\\2\end{pmatrix}t\right]s$$

Observe that the parameter curves are Bézier curves. So $B(t) = S(s_0, t) = \sum \overline{\mathbb{P}}_j B_{j,p}(t)$ where $\overline{\mathbb{P}}_{j_0} = \sum \mathbb{P}_{i,j_0} B_{i,n}(s_0)$ : control points for the $t$-curves : Bézier curves of degree $n$ evaluated at $\underline{s_0}$

In the same way

$$\overline{B}(s) = S(s, t_0) = \sum_{i=0}^{n} \underbrace{\left(\sum_j \mathbb{P}_{i,j} B_{j,p}(t_0)\right)}_{\widetilde{\mathbb{P}}_{i}} B_{i,n}(s)$$

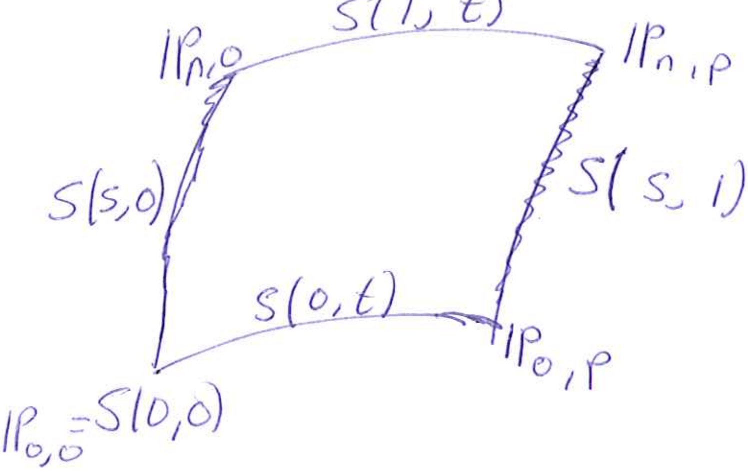
Some properties of Bézier surfaces

i) $S(0,0) = \mathbb{P}_{0,0}$

$S(1,0) = \mathbb{P}_{n,0}$

$S(0,1) = \mathbb{P}_{0,p}$

$S(1,1) = \mathbb{P}_{n,p}$




ii) Convex-Hull Property

$S(s,t) \in CH\{\mathbb{P}_{0,0}, \dots, \mathbb{P}_{n,p}\}$

In the example. $S(s,t)$ contained in the tetrahedron with vertices $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\2\\4\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}$ and $\begin{pmatrix}6\\4\\2\end{pmatrix}$.

iii) Invariance under Affine Transformations

For T affine transformation

$$T\left(\sum\sum \mathbb{P}_{i,j} B_{i,n}(s) B_{j,p}(t)\right) = \sum\sum T(\mathbb{P}_{i,j}) B_{i,n}(s) B_{j,p}(t)$$

<u>Example</u> The transformed of the surface in our example by a rotation of ~~degree~~ 60° anticlockwise around the z-axis is the Bézier surface with control polyhedron

$$\begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 2 & 6 \\ 2 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{array}{cccc} \left(\begin{array}{c|c|c|c} -1.23 & 0.27 & -1.71 & -0.46 \\ 1.87 & 4.46 & 3.23 & 7.19 \\ 3 & 4 & 1 & 2 \end{array}\right) \\ \overline{\mathbb{P}}_{0,0} \quad \overline{\mathbb{P}}_{0,1} \quad \overline{\mathbb{P}}_{1,0} \quad \overline{\mathbb{P}}_{1,1} \end{array}$$

# Derivatives of Bezier Surfaces

$$S(s,t) = \sum \sum \mathbb{P}_{i,j} B_{i,n}(s) B_{j,p}(t)$$
$$= \sum_j \left( \sum_i \mathbb{P}_{ij} B_{i,n}(s) \right) B_{j,p}(t)$$

Differentiating in s we get

$$S_s(s,t) = \sum \left( n \sum_{i=0}^{n-1} (\mathbb{P}_{i+1,j} - \mathbb{P}_{i,j}) B_{i,n-1}(s) \right) B_{j,p}(t)$$
$$= \sum^{n-1} \sum \mathbb{P}_{i,j}^{(1,0)} B_{i,n-1}(s) B_{j,p}(t)$$

Where $\mathbb{P}_{i,j}^{(1,0)} = n(\mathbb{P}_{i+1,j} - \mathbb{P}_{i,j})$

In the same way

$$S_t(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{p-1} \mathbb{P}_{i,j}^{(0,1)} B_{i,n}(s) B_{j,p-1}(t)$$

$$\mathbb{P}_{i,j}^{(0,1)} = p(\mathbb{P}_{i,j+1} - \mathbb{P}_{i,j}) \qquad 0 \leq j \leq p-1$$

Higher order derivatives are done in the same way

$$S^{(\alpha,\beta)}(s,t) = \frac{\partial^{\alpha+\beta} S(s,t)}{\partial^{\alpha} s \, \partial^{\beta} t}$$
$$= \sum_{i=0}^{n-\alpha} \sum_{j=0}^{p-\beta} \mathbb{P}_{i,j}^{(\alpha,\beta)} B_{i,n-\alpha}(s) B_{j,p-\beta}(t)$$

$$\mathbb{P}_{i,j}^{(\alpha,\beta)} = \frac{n!}{(n-\alpha)!} \frac{p!}{(p-\beta)!} \sum_{k=0}^{\alpha} \sum_{h=0}^{\beta} (-1)^k (-1)^h \binom{\alpha}{k} \binom{\beta}{h} \mathbb{P}_{i+\alpha-k,\, j+\beta-h}$$

Example: Partial derivatives of the biquadratic Bezier surface with control points

$P_{0,0}(7,-3,-5)$, $P_{0,1}(7,-2,-6)$, $P_{0,2}(8,-1,-4)$

$P_{1,0}(4,-3,-2)$, $P_{1,1}(5,-1,-4)$, $P_{1,2}(4,0,-3)$

$P_{2,0}(1,-4,1)$, $P_{2,1}(0,2,0)$, $P_{2,2}(1,-3,1)$

$S_s(s,t)$ has control points

$$P_{0,0}^{(1,0)} = 2(P_{1,0}-P_{0,0}) = (-6,0,6)$$
$$P_{1,0}^{(1,0)} = 2(P_{2,0}-P_{1,0}) = (-6,-2,6)$$
$$P_{0,1}^{(1,0)} = 2(P_{1,1}-P_{0,1}) = (-4,2,4)$$
$$P_{1,1}^{(1,0)} = 2(P_{2,1}-P_{1,1}) = (-10,-2,8)$$
$$P_{0,2}^{(1,0)} = 2(P_{1,2}-P_{0,2}) = (-8,2,2)$$
$$P_{1,2}^{(1,0)} = 2(P_{2,2}-P_{1,2}) = (-6,-6,8)$$

$$\begin{pmatrix}-6\\0\\6\end{pmatrix}(1-s)(1-t)^2 + \begin{pmatrix}-6\\-2\\6\end{pmatrix}s(1-t)^2 + \begin{pmatrix}-4\\2\\4\end{pmatrix}(1-s)2(1-t)t + 2\begin{pmatrix}-10\\-2\\8\end{pmatrix}s(1-t)t + \begin{pmatrix}-8\\2\\2\end{pmatrix}(1-s)t^2 + \begin{pmatrix}-6\\-6\\8\end{pmatrix}st^2$$

## Subdivision of Bezier surfaces

de Casteljau algorithm: is applied first in one direction and then in the other

- de Casteljau in the t-direction: each row of the control polyhedron $(P_{i_0,j})$ is treated as the control polygon of a Bezier curve in t.

• We execute the de Casteljau algorithm for $t = t_0$

Now we get to surfaces along $S(s, t_0)$

To execute the algorithm for $s = s_0$ we consider the columns of the control polyhedron. We subdivide $S(s,t)$ along the parameter curve $S(s_0, t)$.

Biquadratic Bezier surface with controll points

$\mathbb{P}_{0,0}(2,3,0)$, $\mathbb{P}_{0,1}(2,6,3)$, $\mathbb{P}_{0,2}(2,10,0)$

$\mathbb{P}_{1,0}(6,2,1)$ $\mathbb{P}_{1,1}(6,6,4)$, $\mathbb{P}_{1,2}(6,9,1)$

$\mathbb{P}_{2,0}(10,2,0)$, $\mathbb{P}_{2,1}(10,6,3)$, $\mathbb{P}_{2,2}(10,10,0)$

To evaluate $S(0,5,\ 0,25)$ we subdivide by $t=0,25$ and then by $s=0,5$

$t=0,25$

row 0
$$\begin{cases} (2,3,0) & (2,6,3) & (2,10,0) \\ (2,\ 3{,}75,\ 0.75) & (2,7,2{,}25) & \\ (2,\ 4.56,\ 1.125) & & \end{cases}$$

row 1
$$\begin{cases} (6,2,1) & (6,6,4) & (6,9,1) \\ (6,3,1{,}75) & (6,\ 6.75,\ 3.25) & \\ (6,\ 3.94,\ 2{,}125) & & \end{cases}$$

row 2
$$\begin{cases} (10,2,0) & (10,6,3) & (10,10,0) \\ (10,3,\ 0.75) & (10,7,\ 2.25) & \\ (10,4,\ 1.125) & & \end{cases}$$

We obtain two surfaces with control points:

$(2,3,0)$ $(2,\ 3.75,\ 0.75)$ $(2,\ 4.56,\ 1.125)$

$(6,2,1)$ $(6,\ 3,\ 1.75)$ $(6,\ 3.94,\ 2.125)$

$(10,2,0)$ $(10,3,\ 0.75)$ $(10,4,\ 1.125)$

II) (2, 4.56, 1.125) (2, 7, 2.25) (2,10,0)
(6, 3.94, 2.125) (6, 6.75, 3.25) (6,9,1)
(10, 4, 1.125) (10, 7, 2.25) (10,10,0)

We apply de Casteljau algorithm for s = 0.5 to both surfaces (by columns)

(2, 3, 0) (6, 2, 1) (10, 2, 0)
(4, 2.5, 0.5) (8, 2, 0.5)
(6, 2.25, 0.5)

(2, 3.75, 0.75) (6, 3, 1.75) (10, 3, 0.75)
(4, 3.375, 1.25) (8, 3, 1.25)

(6, 3.19, 1.25)

(2 4.56, 1.125) (6, 3.94, 2.125) (10, 4, 1.125)
(4, 4.25, 1.625) (8, 3.97, 1.625)
$\boxed{(6, 4.11, 1.625)}$

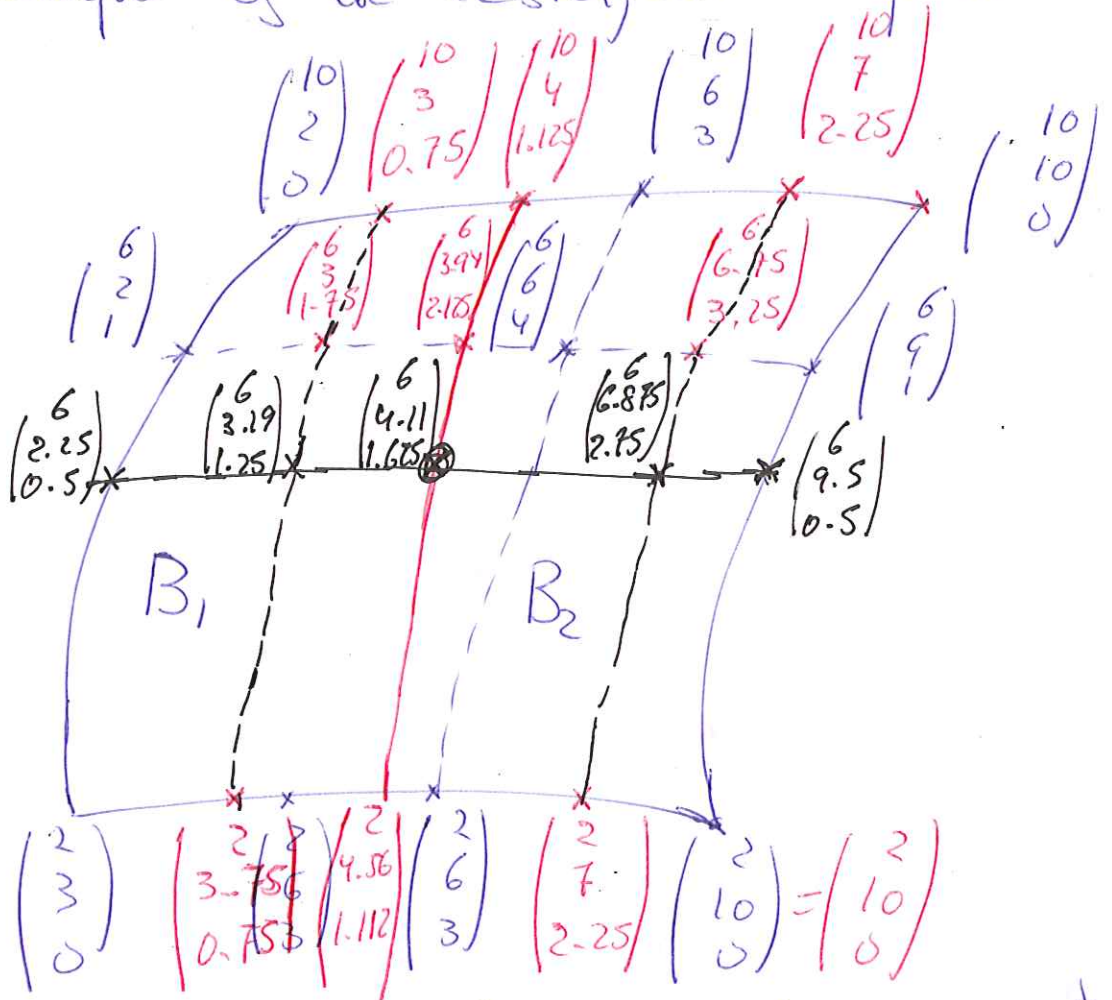
(2, 4.56, 1.125) (6, 3.94, 2.125) (10, 4, 1.125)
(4, 4.25, 1.625) (8, 3.97, 1.625)
$\boxed{(6, 4.11\ 1.625)}$

(2, 7, 2.25) (6, 6.75, 3.25) (10, 7, 2.25)
(4, 6.~~88~~, 2.75) (8, 6.875, 2.75)
(6, 6.875, 2.75)

(2, 10, 0) (6, 9, 1) (10, 10, 0)
(4 9.5 0.5) (8 9.5 0.5)
(6, 9.5 0.5)

Observe that with the algorithm we increase the number of control points for the surface

Situation of the points in the example of de Casteljau algorithm




Observe that the interior points are control points of $S(s,t)$. But the points on the redline belong to $B_1(s,t)$ and $B_2(s,t)$ and $\begin{pmatrix}6\\4.11\\1.625\end{pmatrix}$ is the aproximation of $S(0.5, 0.25)$ that belongs to all the four subdivisions

<u>Complexity</u> Let $S(s,t)$ be a Bezier surface with degrees $m$ (for $s$) and $n$ (for $t$). Then for the subdivision in the $s$-direction we need $\frac{1}{2}m(m+1)(n+1)$ interpolations and for the following subdivision in the $t$-direction $\frac{1}{2}n(n+1)(2m+1)$ interpolations are needed

In total $\boxed{\frac{1}{2}(n+1)(m^2+m+2mn+n) \text{ interpolations}}$