Black hole geometry

Metric & Riemann Tensors in Black Holes, by Göran Bergqvist

Special relativity

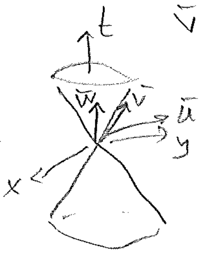
Spacetime = Minkowskispace = $(\mathbb{R}^4, \eta)$

η = flat indefinite (Lorentz) metric

$\bar{v} = (t, x, y, z) \Rightarrow \eta(\bar{v}, \bar{v}) = c^2t^2 - x^2 - y^2 - z^2$, $(\eta_{ij}) = \begin{pmatrix} c^2 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

[often c=1]

Light cone



$\eta(\bar{v}, \bar{v}) = 0$: $\bar{v}$ lightlike (null)

$\eta(\bar{w}, \bar{w}) > 0$: $\bar{w}$ timelike (allowed directions for objects with mass)

$\eta(\bar{u}, \bar{u}) < 0$: $\bar{u}$ spacelike (not allowed)

General relativity

Einstein 1915

Spacetime (M, g), g indef. metric, tangent space = Minkowski

R_{abcd} = Riemann tensor

$R_{ab} = R_a{}^c{}_{bc}$ = Ricci tensor (contraction)

$R = R_c{}^c$ = scalar curvature (contraction)

$R_{abcd} = W_{abcd} + \{\text{expr. in } R_{ab}\}$, W_{abcd} = Weyl curv. tensor

$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$ = Einstein tensor

Einstein field eq. $G_{ab} \; (-\Lambda g_{ab}) = k \cdot T_{ab}$

Λ: cosmological constant; k: constant; T_{ab}: energy-momentum tensor (all physics except gravitation)

If $\Lambda = 0$ $T_{ab} = 0 \Rightarrow G_{ab} = 0 \Leftrightarrow R_{ab} = 0$ Einst. vacuum eq. (system of 10 non-lin. PDE's in components of metric)

In dim ≤ 3, $W_{abcd} = 0$ so $R_{ab} = 0 \Rightarrow$ no curvature at all

dim ≥ 4, $R_{ab} = 0 \not\Rightarrow R_{abcd} = 0$, can have $W_{abcd} \neq 0$

Solutions to $R_{ab} = 0$ in 4 dim

1. Minkowski spacetime. x, y, z replaced by spherical coord r, θ, φ $\Rightarrow$ $(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$

2. Schwarzschild spacetime (1916) $(g_{ij}) = \begin{pmatrix} 1-\frac{2m}{r} & & & \\ & \frac{-1}{1-\frac{2m}{r}} & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix}$

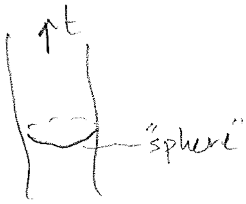
spherically symmetric

$m \sim$ mass ($m = 0 \Rightarrow$ Minkowski) solution outside static, spherically symm. object.

r, θ, φ picture

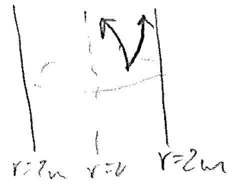
Ricci = 0
Weyl ≠ 0

t, r, θ picture



explains:
{ bending of light
movement of Mercury

If "radius" r of object, $r < 2m$ (≈ 3 km for mass of sun), $r = 2m$ is a coord. singularity (not a real sing: r a bad coord. there), but timelike and null curves (incl. geodesics) starting inside $r < 2m$ cannot reach $r > 2m$ for increasing t



$r = 2m$ is an event horizon
$r < 2m$ is a black hole (r = 0 real singularity)

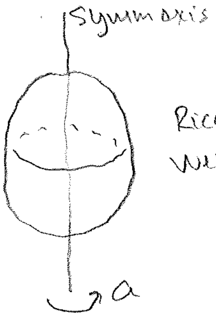
3. Kerr spacetime (1963!)

$$(g_{ij}) = \begin{pmatrix} \frac{\Delta - a^2\sin^2\theta}{\rho^2} & 0 & 0 & \frac{2mar\sin^2\theta}{\rho^2} \\ 0 & -\rho^2/\Delta & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ \frac{2mar\sin^2\theta}{\rho^2} & 0 & 0 & -\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta \end{pmatrix}$$

stationary, axisymmetric, rotating solution to $R_{ab} = 0$

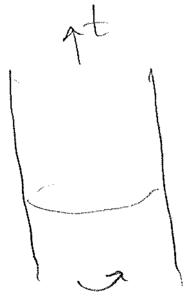
with $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2mr + a^2$

$a = 0 \Rightarrow$ Schwarzschild



Ricci = 0
Weyl ≠ 0

r, θ, φ



t, r, θ

$r = m + \sqrt{m^2 - a^2}$ event horizon
$r < m + \sqrt{m^2 - a^2}$ black hole

Singularity theorems (Penrose, Hawking)
↑ Nobel prize 2020

(note: no symmetry assumed)

I. Universe expanding, Energy cond } ⇒ ∃ singularity at finite time backwards (big bang)

II Gravitational collapse, Energy cond } ⇒ black holes and singularity will form

Black hole uniqueness theorem:

Any stationary vacuum black hole has Kerr geometry
(any black hole should settle to Kerr)