B - Splines

- Polynomial curve defined on the interval $[a,b]$
  i) On each segment of the curve the degree is at most d.
  ii) Sequence of points (the knots) $t_0, \dots, t_m$ such that $t_i \leq t_{i+1}$ $(0 \leq i \leq m-1)$, $t_d = a$, $t_{m-d} = b$

$$\begin{cases} t_0, \dots, t_d, t_{m-d}, \dots, t_m & \text{: end knots} \\ t_{d+1}, \dots, t_{m-d-1} & \text{: interior knots} \end{cases}$$

We need basis functions $N_{i,d}(t)$ defined by

B-S $N_{i,0}(t) = \begin{cases} 1 & t \in [t_i, t_{i+1}) \\ 0 & \end{cases}$

$$N_{i,d}(t) = \frac{t - t_i}{t_{i+d} - t_i} N_{i,d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} N_{i+1,d-1}(t)$$

B-spline curve $B(t) = \sum_{i=0}^{n} \mathbb{b}_i N_{i,d}(t)$, with control points $\mathbb{b}_0, \dots, \mathbb{b}_n$.

<u>Example</u> Basis functions for the quadratic B-spline with $\mathbb{b}_0(1,2)$, $\mathbb{b}_1(3,5)$, $\mathbb{b}_2(6,2)$, $\mathbb{b}_3(9,4)$ and knots $t_0 = 2$, $t_1 = 4$, $\underline{t_2 = 5}$, $t_3 = 7$, $\underline{t_4 = 8}$, $t_5 = 10$, $t_6 = 11$. The curve is defined on the interval $[5, 8]$

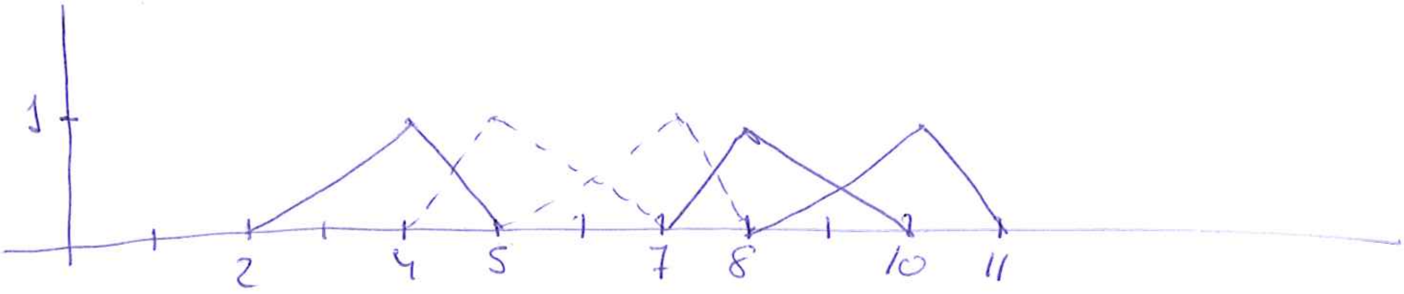
$k = 0$ $\begin{cases} \underline{N_{0,0}}(t) = \{1, \ t \in [2,4) \\ N_{5,0}(t) = \{1 \ \ t \in [10,11) \end{cases}$

$$N_{0,1}(t) = \frac{t-t_0}{t_1-t_0} N_{0,0}(t) + \frac{t_2-t_1}{t_2-t_1} N_{1,0}(t)$$

$$= \frac{t-2}{2} N_{0,0}(t) + (5-t) N_{1,0}(t) \quad \begin{cases} \frac{t-2}{2} & t \in [2,4) \\ 5-t & t \in [4,5) \end{cases}$$

$$N_{3,1}(t) = \frac{t-t_3}{t_4-t_3} N_{3,0}(t) + \frac{t_5-t}{t_5-t_4} N_{4,0}(t) = \begin{cases} t-7 & t \in [7,8) \\ \frac{10-t}{2} & t \in [8,10) \end{cases}$$

$$N_{4,1}(t) = \frac{t-t_4}{t_5-t_4} N_{4,0}(t) + \frac{t_6-t}{t_6-t_5} N_{5,0}(t) = \begin{cases} \frac{t-8}{2} & t \in [8,10) \\ 11-t & t \in [10,11) \end{cases}$$



And for degree $d = 2$.

$$N_{0,2}(t) = \frac{t-t_0}{t_2-t_0} N_{0,1}(t) + \frac{t_3-t_1}{t_3-t_1} N_{1,1}(t)$$

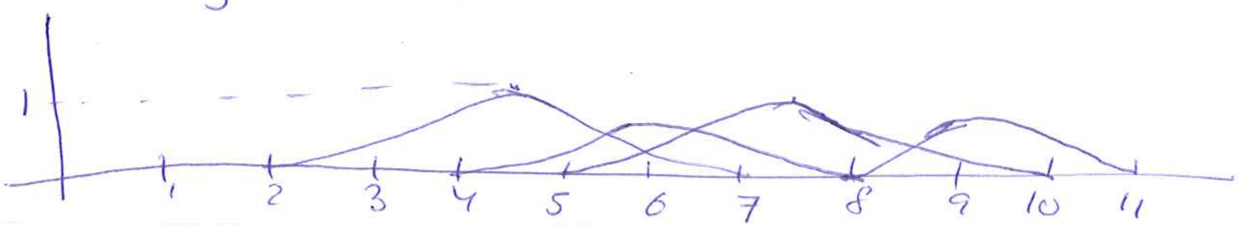
$$= \frac{1}{6}(t-2)^2 N_{0,0} + \frac{1}{3}(-2t^2+18t-38) N_{1,0}(t) + \frac{1}{6}(7-t)^2 N_{2,0}(t)$$

$$N_{1,2}(t) = \frac{t-t_1}{t_3-t_1} N_{1,1}(t) + \frac{t_4-t}{t_4-t_2} N_{2,1}(t) =$$

$$= \frac{1}{3}(t-4)^2 N_{1,0}(t) + \frac{1}{3}(-t^2+12t-34) N_{2,0}(t) + \frac{1}{3}(8-t)^2 N_{3,0}(t)$$

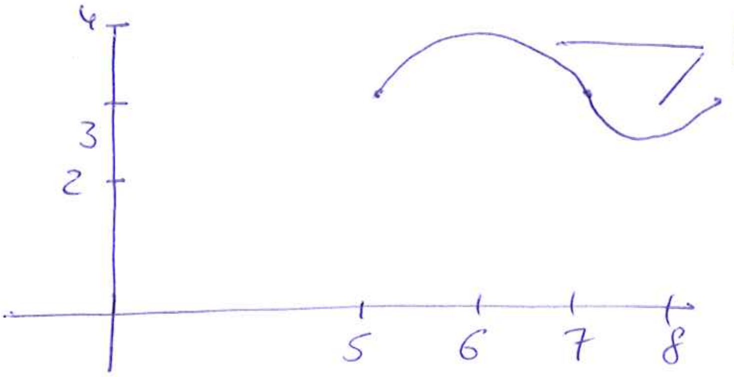
$$N_{3,2}(t) = \frac{t-t_3}{t_3-t_2} N_{3,1}(t) + \frac{t_6-t}{t_6-t_4} N_{4,1}(t)$$

$$= \frac{1}{3}(t-7)^2 N_{3,0}(t) + \frac{1}{3}(-t^2+18t-79) N_{4,0}(t) + \frac{1}{3}(11-t)^2 N_{5,0}(t)$$



$$B(t) = \begin{pmatrix}1\\2\end{pmatrix} N_{0,2}(t) + \begin{pmatrix}3\\5\end{pmatrix} N_{1,2}(t) + \begin{pmatrix}6\\2\end{pmatrix} N_{2,2}(t) + \begin{pmatrix}9\\4\end{pmatrix} N_{3,2}(t)$$

$$= \begin{cases} \begin{pmatrix}1\\2\end{pmatrix}\frac{1}{6}(7-t)^2 + \begin{pmatrix}3\\5\end{pmatrix}\frac{1}{3}(-t^2+12t-34) + \frac{1}{6}\begin{pmatrix}6\\2\end{pmatrix}(t-5)^2 & t \in [5,7) \\ \begin{pmatrix}3\\5\end{pmatrix}\frac{1}{3}(8-t)^2 + \frac{1}{3}\begin{pmatrix}6\\2\end{pmatrix}(-2t^2+30t-110) + \frac{1}{3}(t-7)^2\begin{pmatrix}9\\4\end{pmatrix} & t \in [7,8] \end{cases}$$



Minimal number of knots : $m = n+d+1$

Properties i) $N_{i,k}(t) > 0 \quad t \in (t_i, t_{i+k+1})$

ii) $N_{i,k}(t) = 0 \quad t \notin (t_i, t_{i+k+1})$

iii) $N_{i,k}(t)$ piecewise polynomial

(iv) $\sum_{j=r-k}^{r} N_{j,k}(t) = 1 \quad t \in [t_r, t_{r+1})$

v) Local control: To evaluate the B-spline $B(t) = \sum b_i N_{i,d}(t)$ at $t \in [t_r, t_{r+1})$ $(d \leq r \leq m-d-1)$ we need to evaluate $N_{r-d,d}(t), \ldots, N_{r,d}(t)$.

vi) Again $B(t) \in CH\{b_0, \ldots, b_r\}$ for $t \in [t_r, t_{r+1})$

vii) If T affine transformation
$$T\left(\sum b_i N_{i,d}(t)\right) = \sum T(b_i) N_{i,d}(t)$$

viii) Continuity: If p_i is the multiplicity of the breakpoint $t = u_i$ Then $B(t)$ is $\mathcal{C}^{d-p_i}$ (at least) at $t = u_i$
$\mathcal{C}^\infty$ elsewhere

(x) Derivatives

If $\mathbb{B}(t) = \sum \mathbb{b}_i N_{i,d}(t)$ is a B-spline curve

then $\dot{\mathbb{B}}(t) = \sum_{i=0}^{n-1} \mathbb{b}_i^{(1)} N_{i,d-1}^{(1)}(t)$

where $\mathbb{b}_i^{(1)} = d \dfrac{\mathbb{b}_{i+1} - \mathbb{b}_i}{t_{i+d+1} - t_i}$ and

$N_{i,d-1}^{(1)}$ are the degree $d-1$ basis functions defined over the knots $\{t_0, \ldots, t_{m-1}\}$

Non-uniform rational B-splines (NURBS)

are rational curves obtained from (integral) B-splines

by $\mathbb{B}(t) = \dfrac{\sum w_i \mathbb{b}_i N_{i,d}(t)}{\sum w_i N_{i,d}(t)}$

with knots $\{t_0, \ldots, t_m\}$, $w_0, \ldots w_n$: weights, $\mathbb{b}_0, \ldots, \mathbb{b}_n$ control points and $N_{i,d}$ basis functions over $\{t_0, \ldots, t_m\}$ (If $w_i = 0$ then we replace $w_i \mathbb{b}_i$ by $\mathbb{b}_i$)

B-spline surface

$$S(s,t) = \sum_0^n \sum_0^p \mathbb{P}_{i,j} N_{i,d}(s) N_{j,e}(t)$$

$$(s,t) \in [s_d, s_{m-d}] \times [t_e, t_{q-e}]$$

with $\{\mathbb{P}_{i,j}\}$ control points and $\{s_0, \ldots, s_m\}$ and $\{t_0, \ldots, t_q\}$ knots for the B-splines

$n = m-d-1$, $p = q-e-1$.

NURBS surface is defined as above with

$$S(s,t) = \frac{\sum^{n}\sum^{p} w_{i,j} \mathbb{P}_{i,j} N_{i,d}(s) N_{j,e}(t)}{\sum \sum w_{i,j} N_{i,d}(s) N_{j,e}(t)}$$, w_{ij}: weights.

Properties of B-splines surfaces

Local control : Each segment is determined by

- $(d+1)\times(e+1)$ mesh of points

$s \in [s_\sigma, s_{\sigma+1})$ and $t \in [t_r, t_{r+1})$ $\left(\begin{array}{l} d \leq \sigma \leq m-d-1 \\ e \leq r \leq q-e-1 \end{array}\right)$

$$- S(s,t) = \sum_{i=\sigma-d}^{\sigma} \sum_{j=r-e}^{r} \mathbb{P}_{i,j} N_{i,d}(s) N_{j,e}(t)$$

- Still invariant under affine transformations

Properties of NURBS

Local control : As for B-splines and with the same notation as B-splines surfaces

$$S(s,t) = \frac{\sum_{\sigma-d}^{\sigma} \sum_{r-e}^{r} \mathbb{P}_{i,j} w_{ij} N_{i,d}(s) N_{j,e}(t)}{\sum_{\sigma-d}^{\sigma} \sum_{r-e}^{r} w_{ij} N_{i,d}(s) N_{j,e}(t)}$$

- Invariant under affine transformations
- (Also invariant under projective transformations)

Derivatives

$$S(s,t) = \sum \mathbb{P}_{ij} N_{i,d}(s) N_{j,e}(t)$$

$$S^{(1,0)}(s,t) = \left(\frac{\partial S(s,t)}{\partial s}\right) = \sum_{0}^{n-1} \sum_{0}^{p} \mathbb{P}_{ij}^{(1,0)} N_{i,d-1}^{(1)}(s) N_{j,e}(t)$$

with $\mathbb{P}_{(i,j)}^{(1,0)} = d \dfrac{\mathbb{P}_{(i+1,j)} - \mathbb{P}_{ij}}{s_{i+d+1} - s_{i+1}}$ and $N_{i,d-1}^{(1)}$ basis functions on the knot vector $\{s_0, \dots, s_{m-1}\}$ $(m = n+d+1)$

And $S^{(0,1)}(s,t) = \left(\dfrac{\partial S(s,t)}{\partial t}\right) = \sum_{0}^{n} \sum_{0}^{p-1} \mathbb{P}_{ij}^{(0,1)} N_{i,d}(s) N_{j,e}^{(1)}(t)$

with $\mathbb{P}_{i,j}^{(0,1)} = p \dfrac{\mathbb{P}_{i,j+1} - \mathbb{P}_{i,j}}{t_{j+e+1} - t_{j+1}}$

$N_{j,e}^{(1)}(t)$: basis functions on $\{t_0, \dots, t_{q-1}\}$

Example of construction of surfaces:

Ruled surfaces: from curve $B(s)$ to curve $C(s)$ joined by lines at corresponding points for the parameter s.

if we consider NURBS curves on the knots $\{s_0, \dots s_m\}$ of degree d $\quad B(s) = \dfrac{\sum \mathbb{b}_i u_i N_{i,d}(s)}{\sum u_i N_{i,d}(s)}$, $C(s) = \dfrac{\sum c_j w_j N_{j,d}(s)}{\sum w_j N_{j,d}(s)}$

we construct the NURBS surface, linear in t, by

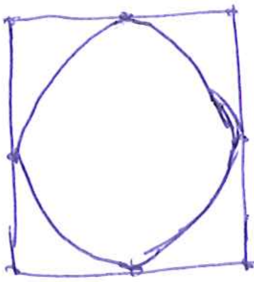
$$S(s,t) = \frac{\sum_{0}^{n} \sum_{0}^{1} v_{i,j} \mathbb{P}_{ij} N_{i,d}(s) N_{j,1}(t)}{\sum v_{ij} N_{i,d}(s) N_{j,1}(t)}$$

$\mathbb{P}_{i,0} = \mathbb{b}_i$, $\quad \mathbb{P}_{i,1} = c_i \quad \{s_0, \dots, s_m\}$, $\{0, 0, 1, 1\}$

$u_{i,0} = u_i \quad v_{i,1} = w_i \quad m = n+d+1$

– Rotation Surfaces

I) We can express a circle as the NURBS with the following control points $\mathbb{b}_0=(1,0)=\mathbb{b}_6$, $\mathbb{b}_1=(1,1)$, $\mathbb{b}_2=(-1,1)$, $\mathbb{b}_3(-1,0)$, $b_4(-1,-1)$, $b_5(1,-1)$ and knot vector $0,0,0,1/4,1/2,1/2,3/4,1,1,1$ and weight $1,1/2,1/2,1,1/2,1/2,1$



- We can use this to obtain the rows (circles of rotation) in the control polyhedron of the NURBS rotation surface

Consider a NURBS on the plane XZ (with always positive x) $\mathbb{B}(s)=\dfrac{\sum_{i=0}^{n}\mathbb{P}_i u_i N_{i,d}(s)}{\sum u_i N_{i,d}(s)}$

with knots $\{s_0,s_1,\dots,s_m\}$ $(m=n+d+1)$

Observe that $\mathbb{P}_i=\begin{pmatrix}x_i\\0\\z_i\end{pmatrix}$, $x_i>0$

The rotation circle on the plane $z=0$ has knots $\{0,0,0,1/4,1/2,1/2,3/4,1,1,1\}$, control points $\mathbb{C}_0(1,0,0)=\mathbb{C}_6$, $\mathbb{C}_1(1,1,0)$, $\mathbb{C}_2(-1,1,0)$, $\mathbb{C}_3(-1,0,0)$, $\mathbb{C}_4(-1,-1,0)$ $\mathbb{C}_5(1,-1,0)$ and weights $1,1/2,1/2,1,1/2,1/2,1$

So the $\overline{\mathbb{P}}_{i,0}=\begin{pmatrix}x_i\\0\\z_i\end{pmatrix}$, $\overline{\mathbb{P}}_{i,1}=\begin{pmatrix}x_i\\x_i\\z_i\end{pmatrix}$, $\overline{\mathbb{P}}_{i,2}=\begin{pmatrix}-x_i\\x_i\\z_i\end{pmatrix}$, $\overline{\mathbb{P}}_{i,3}=\begin{pmatrix}-x_i\\0\\z_i\end{pmatrix}$, $\overline{\mathbb{P}}_{i,4}=\begin{pmatrix}-x_i\\-x_i\\z_i\end{pmatrix}$

and $\overline{\mathbb{P}}_{i,5}=\begin{pmatrix}x_i\\-x_i\\z_i\end{pmatrix}$ (of course $\overline{\mathbb{P}}_{i,6}=\overline{\mathbb{P}}_{i,0}$) $0\le i\le n$.

[We specify the control points by different meridians; 6 columns in the matrix of the control polyhedron]

Finally the weights are

$$w_{i,j} = \{u_i, \tfrac{1}{2}u_i, \tfrac{1}{2}u_i, u_i, \tfrac{1}{2}u_i, \tfrac{1}{2}u_i, u_i\}$$
$$0 \leq i \leq n.$$

of course knots for the meridians are

$$s_0, s_1, \dots, s_m$$

and knots for the parallels are

$$0, 0, 0, 1/4, 1/2, 1/2, 3/4, 1, 1, 1$$

Observe that the different radii of the parallels are obtained by scalating the control points of the parallels (t-curves, dus NURBS circles)