De Casteljau Algorithm (1959)


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For each new column we have the following “deCasteljau” map

Input: array **P**[0:n] of n+1 points and real number u in [0,1] **Output:** point on curve, **C**(u)
Working: point array **Q**[0:n]

for i := 0 **to** n **do**
Q[i] := **P**[i]; // save input

for k := 1 **to** n **do**
for i := 0 **to** n - k **do**

Q[i] := (1 - u)**Q**[i] + u **Q**[i + 1]; **return** **Q**[0];

Recursively (this algorithm is very inefficient)

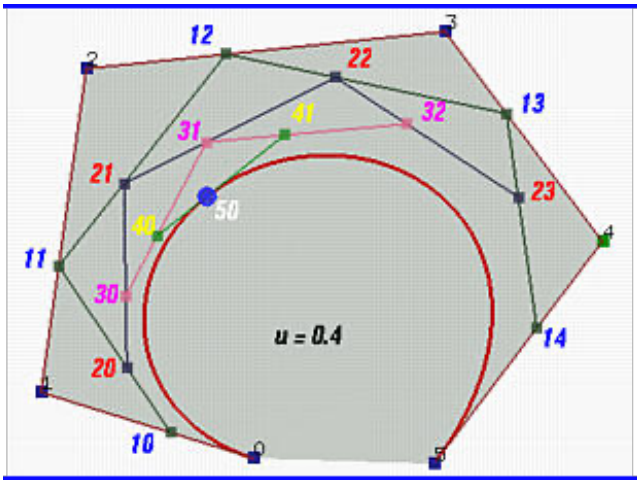
function **deCasteljau**(i,j) **begin**

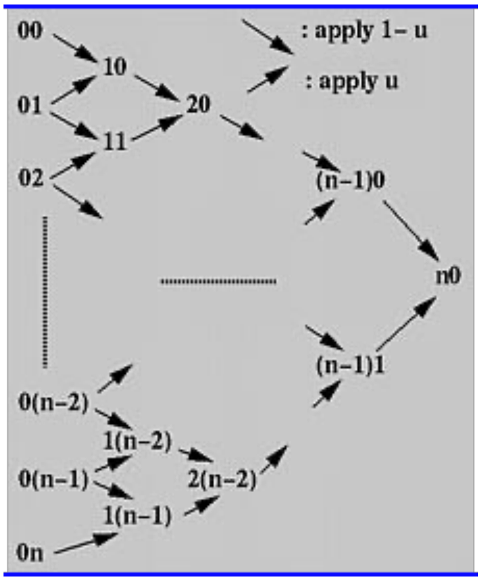
if i = 0 **then return** $\mathbf{P}_{0,j}$

else
return (1-u)* **deCasteljau**(i-1,j) + u* **deCasteljau**(i-1,j+1)

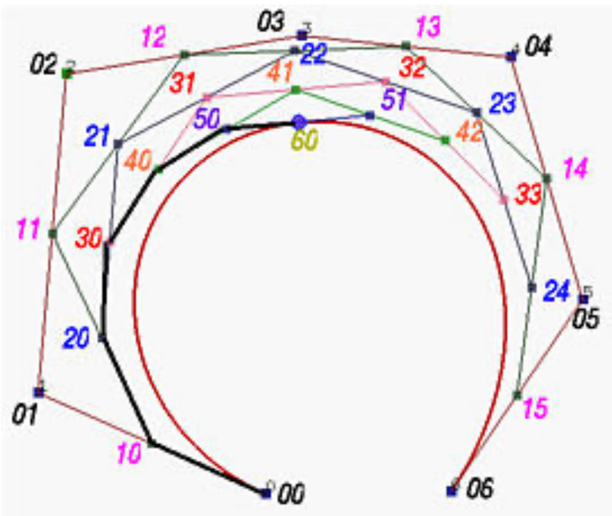
end

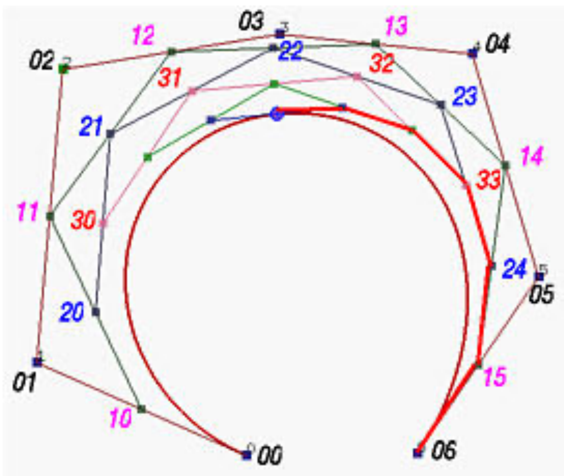
Geometric Images of de Casteljau geometric and computational algorithms, using the recurrence law of Berstein's polynomials





Notice that the points P_{00}, P_{10}, …, Pn0 = B(t_0) are control points for the arc of the Bézier curve from P_0 to B(t_0). In the same way Pn0 = B(t_0), $P_{(n-1)1}$, …., $P_{1(n-1)}$, P_n are control points of the arc of the curve from B(t_0) to P_n.





Observation: One can prove (not so easy) that **one can approximate the curve as closely as wanted using a control polygon**.

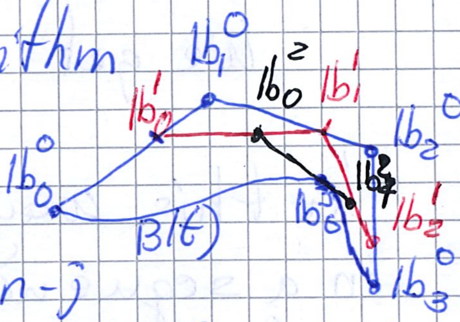
The de Casteljau Algorithm

de Casteljau algorithm is a method to evaluate a Bézier curve at any point. It allows to subdivide the curve, using the good recurrence properties of Bernstein polynomials, and so rendering the curve; in fact as a polygonal curve formed by the sequence of control polygons (linear ones)

Let $B(t) = \sum_{i=0}^{n} b_i \, B_{i,n}(t) \quad , t \in [0,1]$

$\left(B_{i,n}(t) = \begin{cases} 0 & \text{if } i \notin \{0, \dots, n\} \\ \binom{n}{i} (1-t)^{n-i} t^i \end{cases} \right)$

We have the recurrence algorithm



for $t \in [0,1]$ $\begin{cases} b_i^0 = b_i \\ \underline{b_i^j = (1-t)\, b_i^{j-1} + t\, b_{i+1}^{j-1}} \end{cases}$

$j = 1, 2, \dots, n$ and $i = 0, \dots, n-j$

$$\begin{array}{ccccccccccc}
b_0^0 & & b_1^0 & \dots & b_2^0 & \dots & & b_{n-2}^0 & & b_{n-1}^0 & & b_n^0 \\
& b_0^1 & & b_1^1 & & & & & b_{n-2}^1 & & b_{n-1}^1 & \\
& & b_0^2 & & & & & & & b_{n-2}^2 & & \\
& & & b_0^j & & & & b_{n-j}^j & & & & \\
& & & & b_0^{n-1} & & b_1^{n-1} & & & & & \\
& & & & & b_0^{n} & & & & & &
\end{array}$$

<u>Th</u> Let $B(t) = \sum_{i=0}^{n} b_i \, B_{i,n}(t)$ be a Bézier curve with control pts $b_0, \dots, b_n$. Consider the algorithm above. Then for a parameter $t \in [0,1]$ $B(t) = b_0^n$

To prove it one uses the recurrence property of Berstein polynomials:

$$B_{i,n}(t) = (1-t)B_{i,n-1}(t) + t\, B_{i-1,n-1}(t)$$

So $B(t) = \sum b_i\, B_{i,n}(t) =$

$$= \sum_{i=0}^{n-1} b_i (1-t) B_{i,n-1}(t) + \sum_{i=1}^{n} b_i\, t\, B_{i-1,n-1}(t)$$

$$B(t) = \sum_{i=0}^{n-1} b_i (1-t) B_{i,n-1}(t) + \sum_{i=0}^{n-1} b_{i+1} B_{i,n-1}(t)$$

$$= \sum_{i=0}^{n-1} \underbrace{(b_i(1-t) + b_{i+1}t)}_{b_i^1} B_{i,n-1}(t)$$

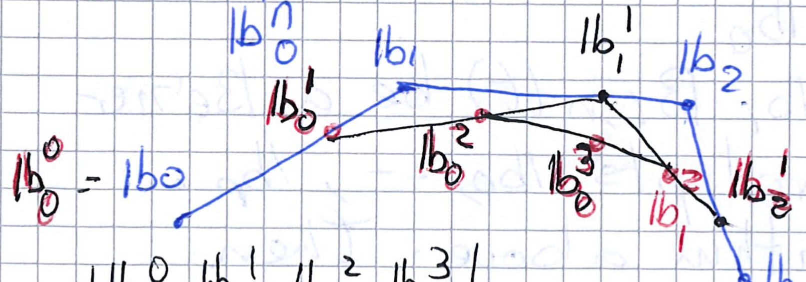
a Bézier curve of deg $n-1$

Applying the same argument for $2, \dots, n$

We get $B(t) = b_0^n$

In this way we can subdivide the curve in a sequence of Bézier curves getting a finer approximation using more control pts without increasing the degree of the polynomials

From de Casteljau algorithm we see that we have got two control polygons: one finishing at $B(t)$, the other starting at $B(t) = b_0^n$



$\{b_0^0, b_0^1, b_0^2, b_0^3\}$

$\{b_0^3, b_1^2, b_2^1, b_3^0\}$

th (Subdivision)

For a general Bézier curve $B(t) = \sum_{i=0}^{n} \mathbb{b}_i B_{i,n}(t)$ the control pts for the two curve segments obtained using de Casteljau algorithm at parameter ~~t~~ α are $\mathbb{b}_0^0, \mathbb{b}_0^1, \ldots, \mathbb{b}_0^n$ for the segment defined over $[0, \alpha]$ and $\mathbb{b}_0^n, \mathbb{b}_1^{n-1}, \ldots, \mathbb{b}_n^0$ for the segment defined over $[\alpha, 1]$

Proof Assume $B(t)$ subdivided at $t = \alpha$

The curve segment $B\_(t)$ is defined over $[0, \alpha]$, reparametrizing $u = \alpha t$ we have

$$B\_(t) = \sum_{i=0}^{n} \mathbb{b}_i B_{i,n}(\alpha t) = \sum_{i=0}^{n} \mathbb{b}_i \left(\sum_{j=0}^{n} B_{i,j}(\alpha) B_{j,n}(t) \right)$$

$$0 \leq \alpha t \leq 1$$

$$= \sum_{j=0}^{n} \left(\sum_{i=0}^{n} \mathbb{b}_i B_{i,j}(\alpha) \right) B_{j,n}(t) = \sum_{j=0}^{n} \mathbb{b}_0^j B_{j,n}(t)$$

- For $B_+(t)$ defined over $[\alpha, 1]$, reparametrize by $t \longmapsto 1-t$.

Piecewise Bézier Curves

First of all a definition for arbitrary definition interval $[t_m, t_M]$.

The Bézier curve with control pts $\mathbb{b}_0, \ldots, \mathbb{b}_n$ defined over the interval $[t_m, t_M]$ is defined by

$$B(t) = \sum_{i=0}^{n} \mathbb{b}_i B_{i,n}\left(\frac{t - t_m}{t_M - t_m}\right)$$

Notice that we have a reparametrization of

$$\tilde{B}(t) = \sum_{0}^{n} \mathbb{b}_i B_{i,n}(t)$$ normalisation

Def Let $I = [a,b]$ $P(t)$ is said to be a piecewise Bézier curve if there is
$a = t_0 < t_1 < \dots < t_{r-1} < t_r = b$ and arbitrary interval Bézier curves $B_j(t)$ defined over $[t_j, t_{j+1}]$ $0 \le j \le r-1$

such that

i) $P(t) = B_j(t)$ $t \in (t_j, t_{j+1})$

ii) $P(t_j) = B_{j-1}(t_j)$ or $B_j(t_j) = P(t_j)$

iii) $P(t_0) = B_0(t_0)$, $P(t_r) = B_{r-1}(t_r)$

The parameters $t_1, \dots, t_{r-1}$ are called breakpts.

The degree of $P(t)$ is the largest degree of the curves $B_j(t)$

Claim 1 $P(t)$ is continuous if for all j
$1 \le j \le r-1$ $\mathbb{b}^{j-1}_{n_{j-1}} = \mathbb{b}^{j}_{0}$ (that is [illegible]

since
$$\underbrace{\lim_{t \to t_j^+} P(t)}_{\mathbb{b}^j_0} = \underbrace{\lim_{t \to t_j^-} P(t)}_{\mathbb{b}^{j-1}_{n_{j-1}}} = P(t_j)$$

where n_{j-1} is the degree of B_{j-1}

In the same way $P(t)$ is C^{k} if for all $p \le k$ and for all j, $1 \le j \le r-1$
$$B^{(p)}_{j-1}(t_j) = B^{(p)}_j(t_j)$$

where $B^{(p)}_j$ is the p-derivative of B_j

or using the chain rule and normalisations
$$\frac{1}{t_j - t_{j-1}} \bar{B}^{(p)}_{j-1}(1) = \frac{1}{t_{j+1} - t_j} \bar{B}^{(p)}_j(0)$$

Example Consider the cubic Bézier curve with control points $P_0(1,0)$, $P_1(2,3)$, $P_2(5,4)$, $P_3(2,1)$. And consider a rotation around the origin anticlockwise through an angle $\pi/4$. The resulting Bezier curve is the Bezier curve with control points $Q_0(0.707, 0.707)$, $Q_1(-0.707, 3.536)$, $Q_2(0.707, 6.364)$ and $Q_3(0.707, 2.121)$

Since we have

$$(q_0, q_1, q_2, q_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 & 2 \\ 0 & 3 & 4 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The de Casteljau Algorithm

Method of evaluating the point on a Bézier curve corresponding to the parameter value $t \in [0,1]$

Very useful to plot (rendering) the curve

Example

Consider a quadratic Bézier curve with controll points $b_0(1.0, 1.0)$, $b_1(8.0, 6.0)$ and $b_2(12.0, 2.0)$

de Casteljau Algorithm gives, for $t = 0.35$

$b_0^0 = b_0 = (1, 1)$, $b_1^0 = b_1 = (8, 6)$ and $b_2^0 = b_2 = (12, 2)$

$b_0^1 = (1-0.35) b_0^0 + 0.35 b_1^0 = (0.65 + 2.80, 0.65 + 2.10) = (3.45, 2.75)$

$b_1^1 = (1-0.35) b_1^0 + 0.35 b_2^0 = (5.2 + 4.2, 3.9 + 0.7) = (9.4, 4.6)$

And finally $B(0.35) = 0.65 b_0^1 + 0.35 b_1^1 = \cancel{(6.11 + 1.61)}$

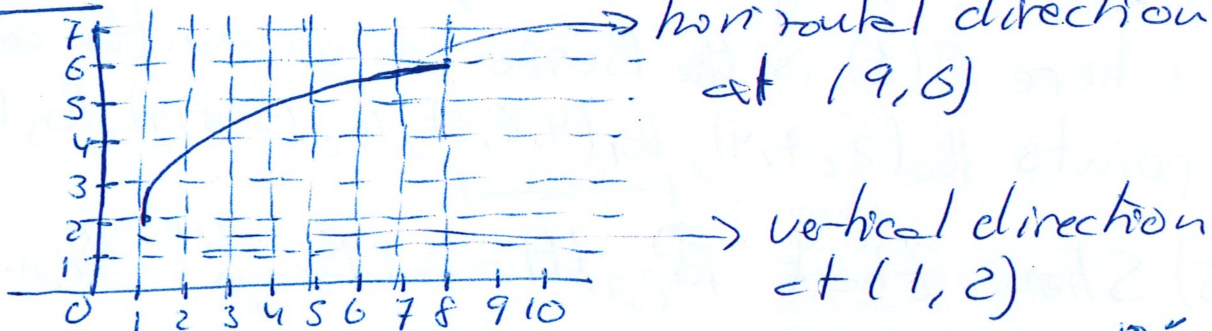
$= (2.24 + 3.29, 1.79 + 1.61)$

$= (5.53, 3.4)$

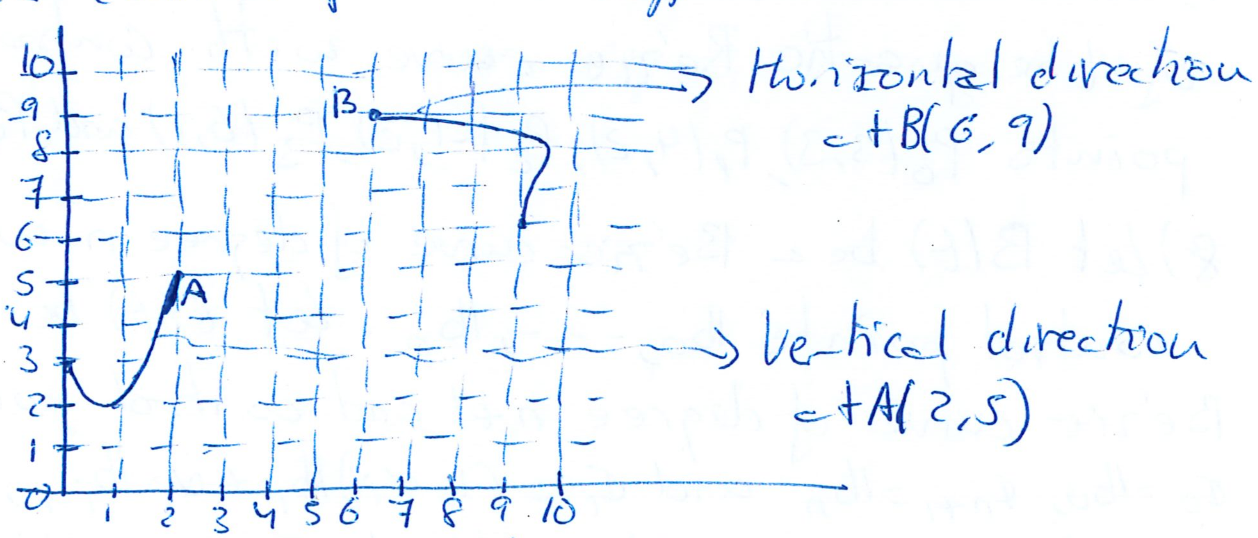
Exercises

Q.1) Write down the parametric form of the quadratic Bézier curve with control points $P_0(-1,5)$, $P_1(2,0)$, $P_2(4,6)$. Evaluate $B(0.65)$ and $B(0.75)$

2a) Suggest plausible control points for the cubic Bézier curve illustrated here



2b) The figure below shows two cubic Bézier curves. Determine the control points and the parametric equation of a cubic Bézier curve joining A and B such that the three cubic Bézier curves form a differentiable curve



3 a) Show that $\int_0^1 B_{i,3}(t) = \frac{1}{4}$, $0 \le i \le 3$

b) Show that $B_{i,n}(t) \geq 0 \quad \forall t \in [0,1]$

4) a) Apply the de Casteljau algorithm to the quartic Bézier curve with control points $P_0(3,3)$, $P_1(4,2)$, $P_2(-1,0)$, $P_3(6,1)$ and $P_4(8,5)$ to evaluate $B(0.65)$

b) Draw the control polygon and the convex hull of the Bézier curve above

c) Determine by de Casteljau algorithm $B(0.3)$, where $B(t)$ is the Bézier curve with control points $\mathbb{b}_0(2,7,4)$, $\mathbb{b}_1(4,6,5)$, $\mathbb{b}_2(5,8,4)$, $\mathbb{b}_3(3,5,3)$.

5) Show that $B'_{i,n}(t) = n(B_{i-1,n}(t) - B_{i,n-1}(t))$

6) Convert the parametric curve
$\gamma(t) = (2-3t-4t^2+7t^3, -4+8t-5t^3)$ into Bézier form.

7) Determine the first and second derivatives of the quartic Bézier curve with control points $P_0(3,3)$, $P_1(4,2)$, $P_2(-1,0)$, $P_3(6,1)$ and $P_4(8,5)$

8) Let $B(t)$ be a Bézier curve of degree n with control points $\mathbb{b}_0, \dots, \mathbb{b}_n$. Let $C(t)$ be the Bézier curve of degree $n+1$ and control points $c_0 = \mathbb{b}_0$, $c_{n+1} = \mathbb{b}_n$ and $c_i = (1-\alpha_i)\mathbb{b}_i + \alpha_i \mathbb{b}_{i-1}$, with $\alpha_i = \frac{i}{n+1}$, $1 \le i \le n$. Show that $C(t) = B(t)$, $\forall t \in [0,1]$.
$C(t)$ is the <u>degree raising</u> of $B(t)$. By degree raising we increase the number of control points to gain freedom of designing curves.