Differential Geometry & Computational Mathematics

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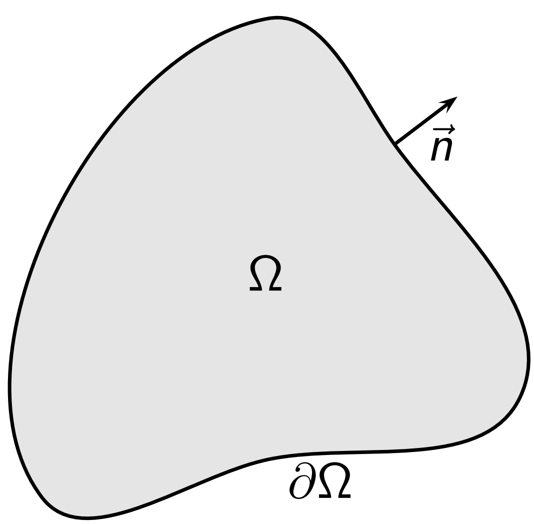
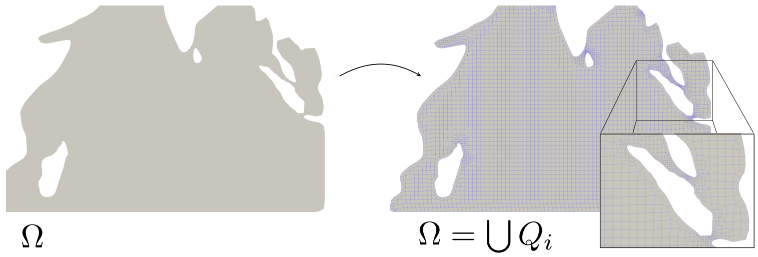
These notes are examples of applications of differential geometry to computational mathematics that Andrew works with daily. According to Andrew differential geometry is an important mathematical tool in the fields of numerical analysis and computational physics. The reason is two-fold.

The first reason is due to the mathematical modelling of a physical problem, e.g., how water waves propagate in the Earth's oceans. The governing equations in such applications are typically built from conservation laws. You can think of the physical principal that "Matter is neither created nor destroyed, it only changes form" and, using a theorem from Emmy Noether, translate this into a mathematical equation regarding the conservation of mass. Importantly, the conservation laws depend on a flux, which we can think of heuristically as "how is information entering or leaving a particular domain". Such a domain is given below where we need to know how information, via the physical flux, passes through the boundary of the domain in the normal direction. It is this normal vector (i.e. contravariant vector) information that differential geometry gives us.



The second reason is from a numerical analysis standpoint. The conservation laws that mathematically model physical phenomena are often too complicated to solve by hand (expect in very simplified cases). We can use mathematical analysis to know that solutions exist but cannot represent this solution in a closed form (say with analytic functions). So we turn to the power of computers to develop algorithms and discretely represent an approximation to the solution of conservation laws. A key to this technology is how to represent a complicated physical domain, like the wing of an airplane or the coastline of an ocean on Earth. One way to turn this difficult problem into a simpler problem, we subdivide the entire domain into a set of non-overlapping quadrilaterals. The example below demonstrates this process for a domain of the Indian Ocean (where the coastline is approximated using splines constructed from real world data).



Once we have this set of non-overlapping quadrilaterals we apply differential geometry tools to gather information about fluxes in the normal directions on each of these smaller pieces of the domain.

Depending on the application this process of dividing a large domain into a set of smaller sub-domains is non-trivial. However, once it is accomplished our differential geometry tools can be applied in two or three spatial dimensions to encode information about normal vectors, the orientation of the surface, and how they apply to a physical flux. Below are two more 3D examples of such subdivisions: One is a domain around Io, a moon of Jupiter, where particular focus is given to the North and South poles of Io as we want to model its magnetosphere. The second is a special kind of spherical shell geometry useful for applications like atmospheric modelling.