# Lösningar, TATA77, 2023-01-04

1. $y''(t) - 2y'(t) + 2y(t) = 2e^t \cos t$, $t \geq 0$, $y(0) = 1$, $y'(0) = 2$.

Enkelsidig laplacetransform ger $(Y = \mathcal{L}_+ y)$:

$$s^2 Y(s) - 1s - 2 - 2(sY(s) - 1) + 2Y(s) = 2\frac{s-1}{(s-1)^2+1},$$

$$(s^2 - 2s + 2)Y(s) = \frac{2(s-1)}{s^2-2s+2} + s,$$

$$Y(s) = \frac{2(s-1)}{(s^2-2s+2)^2} + \frac{s}{s^2-2s+2} = \frac{2(s-1)}{((s-1)^2+1)^2} + \frac{(s-1)+1}{(s-1)^2+1}, \quad \text{Re } s > 1.$$

Inverstransform med tabell ger $y(t)$,

Svar: a) $\frac{2(s-1)}{(s^2-2s+2)^2} + \frac{s}{s^2-2s+2}$, $\text{Re } s > 1$.

b) $e^t((t+1)\sin t + \cos t)$, $t \geq 0$.

2. $u(t) + \int_{-\infty}^{t} e^{r-t} u(r)\,dr = e^{-|t|}$, $t \in \mathbb{R}$.

Integralen $= (f * u)(t)$, där $f(t) = e^{-t}\chi(t)$, $t \in \mathbb{R}$, så fourier-transform ger: $\hat{u}(\omega) + \frac{1}{1+i\omega}\hat{u}(\omega) = \frac{2}{1+\omega^2}$, $\omega \in \mathbb{R}$.

Så $\hat{u}(\omega) = \frac{2(1+i\omega)}{(2+i\omega)(1+\omega^2)} = \frac{2\cancel{(1+i\omega)}}{(2+i\omega)\cancel{(1+i\omega)}(1-i\omega)} = \frac{2/3}{2+i\omega} + \frac{2/3}{1-i\omega}$.

$\frac{2}{3}e^{-2t}\chi(t) + \frac{2}{3}e^t\chi(-t)$ har rätt transform (men $u$ är kontinuerlig), så:

Svar: $u(t) = \begin{cases} \frac{2}{3}e^{-2t}, & t \geq 0, \\ \frac{2}{3}e^t, & t \leq 0. \end{cases}$

3. a) $u(t) = |t^3 - 1| = \begin{cases} t^3-1, & t \geq 1, \\ 1-t^3, & t \leq 1. \end{cases}$ Så $u'(t) = \begin{Bmatrix} 3t^2, & t > 1 \\ -3t^2, & t < 1 \end{Bmatrix} + (0-0)\delta_1(t)$,

och $u''(t) = \begin{Bmatrix} 6t, & t > 1 \\ -6t, & t < 1 \end{Bmatrix} + (3-(-3))\delta_1(t) = \begin{Bmatrix} 6t, & t > 1 \\ -6t, & t < 1 \end{Bmatrix} + 6\delta_1(t)$.

b) $\langle t\delta'(2t), \varphi(t)\rangle = \langle \delta'(2t), t\varphi(t)\rangle = \frac{1}{2}\langle \delta'(t), \frac{t}{2}\varphi(\frac{t}{2})\rangle =$

$= -\frac{1}{2}\langle \delta(t), \frac{t}{2}\varphi'(\frac{t}{2})\cdot\frac{1}{2} + \frac{1}{2}\varphi(\frac{t}{2})\rangle = -\frac{1}{2}(0 + \frac{1}{2}\varphi(0)) = \langle -\frac{1}{4}\delta, \varphi\rangle$, $\varphi \in \mathcal{D}(\mathbb{R})$.

Svar: $-\frac{1}{4}\delta$.

c) $tu' = t^2\chi + \delta_{-1} = t^2\chi(t) + \delta(t+1) = t(t\chi(t) - \delta(t+1))$,

så $u'(t) = t\chi(t) - \delta(t+1) + C\delta(t)$, och

$u(t) = \frac{t^2}{2}\chi(t) - \chi(t+1) + C\chi(t) + D$.

Svar: $\frac{t^2}{2}\chi(t) - \chi(t+1) + C\chi(t) + D$, $C, D \in \mathbb{C}$.

4. $u(t) = \cos t, \quad 0 \le t < \pi/2, \quad u(t) = 0, \quad \pi/2 \le t < 2\pi, \quad T = 2\pi \Rightarrow \Omega = \frac{2\pi}{T} = 1.$

$$\hat{u}(n) = \frac{1}{2\pi}\int_0^{\pi/2} (\cos t) e^{-int}\,dt = \frac{1}{2\pi}\int_0^{\pi/2} \frac{1}{2}(e^{it} + e^{-it}) e^{-int}\,dt =$$

$$= \frac{1}{4\pi}\int_0^{\pi/2} (e^{i(1-n)t} + e^{-i(1+n)t})\,dt \overset{\substack{n \ne \pm 1 \\ \downarrow}}{=} \frac{1}{4\pi}\left[ \frac{e^{i(1-n)t}}{i(1-n)} - \frac{e^{-i(1+n)t}}{i(1+n)} \right]_0^{\pi/2} =$$

$$= \frac{1}{4\pi}\left( \frac{ie^{-in\pi/2} - 1}{i(1-n)} - \frac{(-i)e^{-in\pi/2} - 1}{i(1+n)} \right) = \dots = \frac{n - ie^{-in\pi/2}}{2\pi i(n^2 - 1)}, \quad n \ne \pm 1.$$

$$\hat{u}(1) = \frac{1}{4\pi}\int_0^{\pi/2} (1 + e^{-i2t})\,dt = \frac{1}{4\pi}\left[ t - \frac{e^{-i2t}}{2i} \right]_0^{\pi/2} = \frac{1}{8} + \frac{1}{4\pi i},$$

och på samma sätt $\hat{u}(-1) = \frac{1}{8} - \frac{1}{4\pi i}$.

Delsvar: $\left(\frac{1}{8} + \frac{1}{4\pi i}\right)e^{it} + \left(\frac{1}{8} - \frac{1}{4\pi i}\right)e^{-it} + \sum_{n \ne \pm 1} \frac{n - ie^{-in\pi/2}}{2\pi i(n^2-1)} e^{int}$.

Satsen om punktvis konv. ger att u:s f.s. summa i $t=0$ är

$$\frac{u(0+) + u(0-)}{2} = \frac{1+0}{2} = \frac{1}{2}.$$

5. $u(n) - u(n-1) + u(n-2) + \sum_{k=3}^{\infty} 2(-1)^k u(n-k) = 3\chi(n), \quad n \in \mathbb{Z}.$

Summan är $(f * u)(n)$, där $f(n) = \begin{cases} 2(-1)^n, & n \ge 3, \\ 0, & n \le 2, \end{cases}$ så

$$\hat{f}(z) = -\frac{2}{z^3} + \frac{2}{z^4} - \frac{2}{z^5} + \dots = \frac{-2/z^3}{1 - (-1/z)} = \frac{-2}{z^2(z+1)}, \quad |z| > 1.$$

Z-transform ger nu:

$$\hat{u}(z) - \frac{1}{z}\hat{u}(z) + \frac{1}{z^2}\hat{u}(z) - \frac{2}{z^2(z+1)}\hat{u}(z) = \frac{3z}{z-1}, \quad |z| \in R_u \cap ]1, \infty[.$$

$$\frac{z^2(z+1) - z(z+1) + z + 1 - 2}{z^2(z+1)}\hat{u}(z) = \frac{3z}{z-1}, \quad \frac{z^3 - 1}{z^2(z+1)}\hat{u}(z) = \frac{3z}{z-1},$$

$$\hat{u}(z) = \frac{3z}{z-1} \cdot \frac{z^3 + z^2}{z^3 - 1} = z\frac{3(z^3+z^2)}{(z-1)^2(z^2+z+1)} = z\left( \frac{2}{(z-1)^2} + \frac{3}{z-1} + \frac{1}{z^2+z+1} \right) =$$

$$= \frac{2z}{(z-1)^2} + \frac{3z}{z-1} + \frac{z}{z^2+z+1}, \quad \text{med } |z| > 1 \text{ eller } 0 < |z| < 1, \text{ där}$$

$|z| > 1$ måste väljas för att få en lösning.

$$\frac{z}{z^2 + z + 1} = \frac{\frac{2}{\sqrt{3}}(\sin\frac{2\pi}{3})z}{z^2 - 2(\cos\frac{2\pi}{3})z + 1}, \quad \text{så tabell ger:}$$

Svar: $u(n) = \left(2n + 3 + \frac{2}{\sqrt{3}}\sin\frac{2\pi n}{3}\right)\chi(n), \quad n \in \mathbb{Z}.$

6. $u(t) = \sin n$ då $n \le t < n+1$, $n = 0, 1, 2, \dots$, och $u(t) = 0$ då $t < 0$.

$$\hat{u}(s) = \int_{-\infty}^{\infty} u(t) e^{-st}\, dt = \sum_{n=0}^{\infty} \int_n^{n+1} (\sin n) e^{-st}\, dt \overset{s \neq 0}{=}$$

$$= \sum_{n=0}^{\infty} (\sin n) \left[ \frac{e^{-st}}{-s} \right]_n^{n+1} = \sum_{n=0}^{\infty} (\sin n) \frac{e^{-sn} - e^{-s(n+1)}}{s} =$$

$$= \frac{1-e^{-s}}{s} \sum_{n=0}^{\infty} (\sin n)(e^s)^{-n} = / \text{z-transformtabell} /$$

$$= \frac{1-e^{-s}}{s} \cdot \frac{(\sin 1) e^s}{e^{2s} - 2(\cos 1) e^s + 1} \quad \text{då } |e^s| > 1 \text{, dvs då } \operatorname{Re} s > 0.$$

Svar: $\hat{u}(s) = \dfrac{(\sin 1)(e^s - 1)}{s(e^{2s} - 2(\cos 1) e^s + 1)}$, $\operatorname{Re} s > 0$.

7. $u : \mathbb{R} \to \mathbb{C}$ kontinuerlig och $u(t) = \frac{1}{t} + \mathcal{O}\left(\frac{1}{t^2}\right)$ då $t \to \pm\infty$.

Sätt $v(t) = u(t) - \dfrac{t}{t^2+1} = \dfrac{1}{t} + \mathcal{O}\left(\dfrac{1}{t^2}\right) - \dfrac{t}{t^2+1} =$

$$= \frac{t^2+1-t^2}{t(t^2+1)} + \mathcal{O}\left(\frac{1}{t^2}\right) = \frac{1}{t(t^2+1)} + \mathcal{O}\left(\frac{1}{t^2}\right) \quad \text{då } t \to \pm\infty.$$

Detta ger att $v \in L^1(\mathbb{R})$, så $\hat{v}$ är en kontinuerlig funktion.

$$\hat{u}(\omega) = \left(\frac{t}{t^2+1}\right)^{\wedge}(\omega) + \hat{v}(\omega) = -\pi i e^{-|\omega|} \operatorname{sgn} \omega + \hat{v}(\omega),$$

så $\hat{u}$ ges av en funktion som är kontinuerlig utom i origo,

och $\hat{u}(0+) - \hat{u}(0-) = -\pi i + \hat{v}(0) - (-\pi i(-1) + \hat{v}(0)) = \underline{\underline{-2\pi i}}$.