LINKÖPINGS TEKNISKA HÖGSKOLA
Matematiska institutionen

KURSKOD: TATA83
Provkod: TEN1

TENTAMEN I TATA83 (FLERVARIABELANALYS)
2024-03-24 KL 14-19

Inga hjälpmedel tillåtna. Uppgifterna bedöms med 0-3 poäng.
Betyggränser: 8/11/14 poäng ger betyg 3/4/5.

1. (i) Beräkna gränsvärdet $\lim_{(x,y)\to(0,0)} \frac{x\tan(y)}{x^2+y^2}$ eller visa att det inte existerar (2p)

   (ii) Beräkna gränsvärdet $\lim_{(x,y)\to(0,0)} \frac{3x^2y^2-x^2-y^2}{x^2+y^2}$ eller visa att det inte existerar (1p)

2. (i) Bestäm en tangentvektor till kurvan $(x(t), y(t), z(t)) = (6t-11, \frac{t^3}{4}, 2t^2-2t-3)$ $(t \in R)$ i punkten $(x,y,z) = (1,2,1)$ (1p)

   (ii) Bestäm på formen $Ax+By+Cz+D=0$ tangentplanet till nivåytan $xy^2z+2x^2yz^2=8$ i punkten $(x,y,z)=(1,2,1)$ på ytan (1p)

   (iii) Beräkna riktningsderivatan av $f(x,y,z)=yz^2e^x$ i punkten $(0,2,1)$ i den riktning som ges av vektorn $(2,1,2)$ (1p).

3. Bestäm alla lokala maximi- och minimipunkter samt sadelpunkter till funktionen $f(x,y)=x^3+12xy^2+x^2+4y^2$.

4. Bestäm största och minsta värde, om de finns, av funktionen

   $f(x,y)=yx^2+3x-y,$

   på eller innanför triangeln med hörn $A(0,0), B(0,-2), C(2,-2)$.

5. Beräkna dubbelintegralen $\int\int_D (xy-2x^2)\,dxdy$,

   där D ges av olikheterna $0 \le x+y \le 1,\ 1 \le 2x-y \le 3$.

6. Bestäm lösningen $f(x,y)$ till den partiella differentialekvationen

   $4y^2 f''_{xx} - 4yf''_{xy} + f''_{yy} - 2f'_x = 4$

   med bivillkoren $f(x,0)=0$ och $f'_y(x,0)=e^x$, genom att byta till variablerna $u=x+y^2$ och $v=y$.

7. Finn volymen av den kropp som begränsas av ytorna $z=21-4y$ och $z=x^2+y^2$.

1 (i) $\lim_{(x,y)\to(0,0)} \underbrace{\frac{x\tan y}{x^2+y^2}}_{U} = G$

Obs $\left(\frac{0}{0}\right)$; $U\Big|_{x\text{-axeln}} = \frac{x\cdot\tan 0}{x^2+0^2} = \frac{0}{x^2} = 0 \to 0$ då $x\to 0$

$U\Big|_{y\text{-axeln}} = \frac{0\tan y}{0^2+y^2} = \frac{0}{y^2} = 0 \to 0$ då $y\to 0$

Kolla $U\Big|_{y=x} = \frac{x\tan x}{x^2+x^2} = \frac{\tan x}{2x} \to \frac{1}{2}$ då $x\to 0$

$\Rightarrow$ $\underline{G \text{ saknas}}$ ty $0\neq\frac{1}{2}$

(ii) $\lim_{(x,y)\to(0,0)} \underbrace{\frac{3x^2y^2-x^2-y^2}{x^2+y^2}}_{U} = G$

Obs $\left(\frac{0}{0}\right)$, $U\Big|_{x\text{-axeln}} = \frac{3x^2\cdot 0^2-x^2-0^2}{x^2+0^2} = -1 \to -1$ då $x\to 0$

$U\Big|_{y\text{-axeln}} = -1 \to -1$ då $x\to 0$

Betrakta $|U-(-1)| = \frac{3x^2y^2}{x^2+y^2} \leq \frac{3\rho^4}{\rho^2} = 3\rho^2 \to 0$ då $\rho\to 0$

$\Rightarrow \lim_{(x,y)\to(0,0)} U = \underline{-1}$

2 (i) $\underbrace{(x(t), y(t), z(t))}_{\vec{r}(t)} = \left(6t-11,\ \frac{t^3}{4},\ 2t^2-2t-3\right)$

i punkten $(x,y,z) = (1,2,1)$

Bestäm t ~ punkten, Obs $t=2$ (Kontroll: $(1,2,1)$) ok

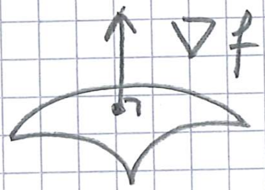
$\vec{r}\,'(t) = \left(6,\ \frac{3t^2}{4},\ 4t-2\right),\ \Rightarrow$

$\vec{r}\,'(2) = (6, 3, 6)$

(ii) nivåyta $\underbrace{xy^2z + 2x^2yz^2}_{f(x,y,z)} = 8$

i punkten $(1,2,1)$

Kontroll: $1\cdot 2^2\cdot 1 + 2\cdot 1^2\cdot 2\cdot 1^2 = 8$ ok!



$$\nabla f = \left(y^2z + 4xyz^2,\ 2xyz + 2x^2z^2,\ xy^2 + 4x^2yz\right)$$

$$\nabla f(1,2,1) = \left(2^2\cdot 1 + 4\cdot 1\cdot 2\cdot 1^2,\ 2\cdot 1\cdot 2\cdot 1 + 2\cdot 1^2\cdot 1^2,\ 1\cdot 2^2 + 4\cdot 1^2\cdot 2\cdot 1\right) = (12, 6, 12) \sim (2,1,2)$$

$\Rightarrow$ tangentplan till ytan i punkten i

$$2(x-1) + 1\cdot(y-2) + 2\cdot(z-1) = 0 \Leftrightarrow$$

$$\underline{2x + y + 2z - 6 = 0}$$

(iii) $f'_{\frac{\vec{v}}{|\vec{v}|}}(0,2,1)$? där $\vec{v} = (2,1,2)$ & $f = y\cdot z^2\cdot e^x$

$$\nabla f = \left(e^x yz^2,\ z^2 e^x,\ 2yze^x\right)$$

$$\nabla f(0,2,1) = \left(e^0\cdot 2\cdot 1^2,\ 1^2\cdot 1,\ 2\cdot 2\cdot 1\cdot e^0\right) = (2, 1, 4)$$

$|\vec{v}| = \sqrt{2^2+1^2+2^2} = 3,\quad \frac{\vec{v}}{|\vec{v}|} = \frac{1}{3}(2,1,2) \quad \Rightarrow$

$$f'_{\frac{\bar{v}}{|\bar{v}|}}(0,2,1) = \nabla f(0,2,1) \cdot \frac{\bar{v}}{|\bar{v}|} = (2,1,4) \cdot \frac{1}{3}(2,1,2) =$$

$$= \frac{1}{3}(4+1+8) = \underline{\frac{13}{3}}$$

(3) Bestäm lokala extrempunkter samt sadelpunkter till $f = x^3 + 12xy^2 + x^2 + 4y^2$

Obs $f \in C^1(R^2)$.

1) Stationära punkter: $\nabla f = \bar{0} \Leftrightarrow \begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases}$

$$\begin{cases} 3x^2 + 12y^2 + 2x = 0 & (1) \\ 24xy + 8y = 0 & (2) \end{cases}$$

(2): $3xy + y = 0 \Leftrightarrow y(3x+1) = 0$

(a) $y = 0 \to (1)$: $3x^2 + 2x = 0 \Leftrightarrow x(3x+2) = 0$

$\Leftrightarrow x_1 = 0,\ x_2 = -\frac{2}{3} \Rightarrow \underline{P_1(0,0) \text{ o } P_2(-\frac{2}{3}, 0)}$

(b) $x = -\frac{1}{3} \to (1)$: $3 \cdot (-\frac{1}{3})^2 + 12y^2 + 2 \cdot (-\frac{1}{3}) = 0$

$\frac{1}{3} + 12y^2 - \frac{2}{3} = 0 \Leftrightarrow 12y^2 = \frac{1}{3} \Leftrightarrow 36y^2 = 1 \Leftrightarrow$

$y = \pm \frac{1}{6} \Rightarrow \underline{P_3(-\frac{1}{3}, \frac{1}{6}),\ P_4(-\frac{1}{3}, -\frac{1}{6})}$

2) P: $Q(h,k) = f''_{xx}(P) \cdot h^2 + 2f''_{xy}(P) \cdot h \cdot k + f''_{yy}(P) k^2$.

$f''_{xx} = (f'_x)'_x = 6x + 2,\ f''_{xy} = (f'_x)'_y = 24y,\ f''_{yy} = 24x + 8$

	f''_{xx}	f''_{xy}	f''_{yy}
P_1	$6\cdot 0+2=2$	$24\cdot 0=0$	$24\cdot 0+8=8$
P_2	$6\cdot\left(-\frac{2}{3}\right)+2=-2$	$24\cdot 0=0$	$24\cdot\left(-\frac{2}{3}\right)+8=-8$
P_3	$6\cdot\left(-\frac{1}{3}\right)+2=0$	$24\cdot\frac{1}{6}=4$	$24\cdot\left(-\frac{1}{3}\right)+8=0$
P_4	$6\cdot\left(-\frac{1}{3}\right)+2=0$	$24\cdot\left(-\frac{1}{6}\right)=-4$	$24\cdot\left(-\frac{1}{3}\right)+8=0$

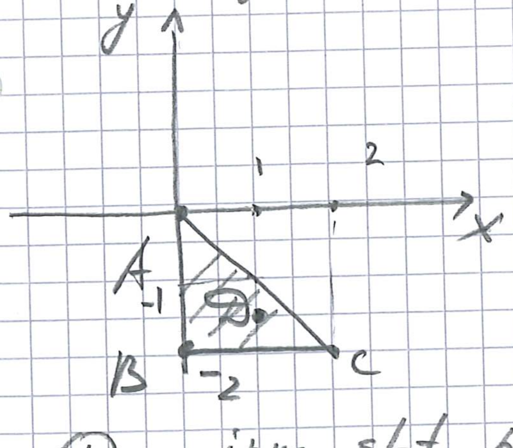
$Q_1(h,k)=2h^2+8k^2$, pos. def. $\Rightarrow$ P_1 är en str. lok. min. punkt

$Q_2(h,k)=-2h^2-8k^2$, neg. def $\Rightarrow$ P_2 är en str. lok. max. punkt.

$Q_3(h,k)=2\cdot 4hk$, indefinit $\Rightarrow$ P_3 är en sadelpunkt.

$Q_4(h,k)=-24hk$, indefinit $\Rightarrow$ P_4 är en sadelpunkt.

(4) $f(x,y)=yx^2+3x-y \longrightarrow$ max/min på triangeln med hörn $A(0,0)$, $B(0,-2)$, $C(2,-2)$



Obs 1) f är kontin. 2) D är kompakt $\Rightarrow$

f har max o min

(1) inre stat. punkter: $\nabla f=\bar{0} \Leftrightarrow \begin{cases} f'_x=0 \\ f'_y=0 \end{cases}$

$$\begin{cases} 2xy+3=0 & (1) \\ x^2-1=0 & (2) \end{cases} \Rightarrow x=\pm 1 \rightarrow (1)$$

$x=1, \quad 2\cdot 1\cdot y+3=0 \Rightarrow y=-3/2 \Rightarrow P_1(1,-3/2)$

$x=-1, \quad 2(-1)\cdot y+3=0 \Rightarrow y=3/2 \Rightarrow P_2(-1,3/2)$

Obs $P_1 \in \operatorname{Int} \mathcal{D}$ (en kandidat)

$P_2 \notin \operatorname{Int} \mathcal{D}$ (ej kandidat)

$f(P_1) = -\frac{3}{2}\cdot 1^2 + 3\cdot 1 - (-3/2) = -\frac{3}{2} + 3 + 3/2 = \underline{\underline{3}}$

(kandidatvärde)

(2) Randen: $\operatorname{Bd}\mathcal{D} = AB \cup BC \cup AC$

$AB: \begin{cases} x=0 \\ y=t, \quad t\in[-2,0] \end{cases}$

$f|_{AB} = f(0,t) = t\cdot 0 + 3\cdot 0 - t = -t = g_1(t), \quad t\in[-2,0]$

en-var: $\quad g_1'(t)=0, \quad t\in(-2,0)$

$-1=0$ inga lösningar.

$BC: \begin{cases} x=t \\ y=-2, \quad t\in[0,2] \end{cases}$

$f|_{BC} = f(t,-2) = -2\cdot t^2 + 3t - (-2) = -2t^2+3t+2 = g_2(t),$
$t\in[0,2]$

en-var: $\quad g_2'(t)=0, \quad t\in(0,2)$

$-4t+3=0 \Leftrightarrow t=3/4 \in (0,2)$ obs

$\Rightarrow P_3(\frac{3}{4},-2)$ en kandidat, $f(P_3) = -2\cdot\left(\frac{3}{4}\right)^2 + 3\cdot\frac{3}{4} + 2 =$

$= -\frac{9}{8} + \frac{9}{4} + 2 = 2 + \frac{9}{8} = \underline{\frac{25}{8}}$ (kandidatvärde)

AC: $\begin{cases} x = t \\ y = -t, \quad t \in [0,2] \end{cases}$

$f\big|_{AC} = f(t,-t) = -t\cdot t^2 + 3t - (-t) = \underbrace{-t^3 + 4t}_{g_3(t)}, \quad t \in [0,2]$

en-var: $g_3'(t) = 0, \quad t \in (0,2)$

$-3t^2 + 4 = 0 \Leftrightarrow t_{1,2} = \pm \frac{2}{\sqrt{3}}$

Obs $\left.\begin{array}{l} \frac{2}{\sqrt{3}} \in (0,2) \\ -\frac{2}{\sqrt{3}} \notin (0,2) \end{array}\right| \Rightarrow P_4\left(\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$ (en kandidat)

$f(P_4) = -\frac{2}{\sqrt{3}}\left(\frac{2}{\sqrt{3}}\right)^2 + 3\cdot\frac{2}{\sqrt{3}} - \left(-\frac{2}{\sqrt{3}}\right) =$

$= \underline{\frac{16}{3\sqrt{3}}}$ (kandidatvärde)

Hörn (A, B, C): $f(A) = \underline{0}$, $f(B) = \underline{2}$,

$f(C) = -8 + 6 + 2 = \underline{0}$ (kandidatvärden)

(3) Lista ut funna kandidatvärde o plocka fram max / min: $\left\{3, \frac{25}{8}, \frac{16}{3\sqrt{3}}, 0, 2\right\}$

Obs $0 < 2 < 3 < \frac{25}{8}$. Jämför $\frac{25}{8} \gtrless \frac{16}{3\sqrt{3}}$ (kvadrera!)

$\frac{625}{64} \gtrless \frac{256}{27}$ eller $16875 > 16384$

$\Rightarrow \left[\begin{array}{ll} \max\limits_{D} f = \frac{25}{8} & \text{antas i } P_3\left(\frac{3}{4}, -2\right) \\ \min\limits_{D} f = 0 & \text{antas i } A \text{ o } C. \end{array}\right.$

$$x(y-2x) = \frac{u+v}{3}$$

5

$$I = \iint_{D} (xy - 2x^2)\,dx\,dy, \quad \text{där}$$

$$D = \{0 \le \underbrace{x+y}_{u} \le 1, \quad 1 \le \underbrace{2x-y}_{v} \le 3\}$$

$$\begin{cases} u = x+y \\ v = 2x-y \end{cases} \quad \frac{\partial uv}{\partial xy} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \quad \left|\frac{d\,xy}{d\,uv}\right| = \frac{1}{3}$$

$$-1-2=-3$$

$$\Rightarrow I = \iint_{D_{uv}} \frac{(u+v)}{3}\cdot(-v)\cdot\frac{1}{3}\,du\,dv =$$

$$D_{uv} = \{0 \le u \le 1, \quad 1 \le v \le 3\}$$

$$-\frac{1}{9}\int_0^1 \left(\int_1^3 (uv+v^2)\,dv\right)du =$$

$$-\frac{1}{9}\int_0^1 \left(\frac{uv^2}{2} + \frac{v^3}{3}\right)\Big|_1^3 du = -\frac{1}{9}\int_0^1 \left(u\cdot\frac{9}{2} + \underbrace{\frac{27}{3}}_{9}\right) - \left(\frac{u}{2} + \frac{1}{3}\right) du$$

$$= -\frac{1}{9}\int_0^1 \left(4u + \frac{26}{3}\right) du = -\frac{1}{9}\left[\cancel{4}^{\,2}\frac{u^2}{\cancel{2}} + \frac{26}{3}u\right]_0^1 =$$

$$= -\frac{1}{9}\left[\left(2+\frac{26}{3}\right) - 0\right] = -\frac{1}{9}\cdot\frac{32}{3} = -\frac{32}{27}$$

6 Bestäm lsg f till PDE

$$4y^2 f''_{xx} - 4y f''_{xy} + f''_{yy} - 2f'_x = 4$$

med $f(x,0) = 0$, $f'_y(x,0) = e^x$.

subst. $u = x + y^2$, $v = y$

$$f'_x = f'_u \cdot \underbrace{u'_x}_{=1} + f'_v \cdot \underbrace{v'_x}_{=0} = f'_u$$

$$f'_y = f'_u \cdot \underbrace{u'_y}_{=2y} + f'_v \cdot \underbrace{v'_y}_{=1} = 2y \cdot f'_u + f'_v$$

$$f''_{xx} = (f'_x)'_x = (f'_u)'_x = f''_{uu}$$

$$f''_{xy} = (f'_x)'_y = (f'_u)'_y = 2y \cdot f''_{uu} + f''_{uv}$$

$$f''_{yy} = (f'_y)'_y = (2y \cdot f'_u + f'_v)'_y =$$

$$= 2 \cdot f'_u + 2y \cdot (f'_u)'_y + (f'_v)'_y = 2 \cdot f'_u +$$

$$+ 2y\left(2y \cdot f''_{uu} + f''_{uv}\right) + \left(2y \cdot f''_{vu} + f''_{vv}\right) =$$

$$= 2f'_u + 4y^2 \cdot f''_{uu} + 4y f''_{vu} + f''_{vv}$$

<u>Ny PDE:</u>

$$4y^2 \cdot \cancel{f''_{uu}} - 4y\left(\cancel{2y \cdot f''_{uu}} + f''_{uv}\right) + \cancel{2f'_u} +$$

$$\cancel{4y^2 \cdot f''_{uu}} + 4y f''_{vu} + f''_{vv} - \cancel{2 \cdot f'_u} = 4$$

$$\boxed{f''_{vv} = 4} \qquad f''_{vv} = (f'_v)'_v \Rightarrow$$

$$f'_v = \int 4 \, dv = 4v + c(u),$$

$$f = \int f'_v \, dv = \int (4v + c(u)) \, dv = \underline{2v^2 + c(u) \cdot v + d(u)}$$

$c(\cdot), d(\cdot)$ är C^2-funktioner.

$$\Rightarrow \quad \underline{f(x,y) = 2y^2 + c(x+y^2)\cdot y + d(x+y^2)}$$

<u>Partikulär lsg:</u>

$$f(x,0) = 0 + \underbrace{c(x)\cdot 0}_{0} + d(x) = 0 \Rightarrow \underline{d=0}$$

$$\Rightarrow \quad f(x,y) = 2y^2 + c(x+y^2)\cdot y$$

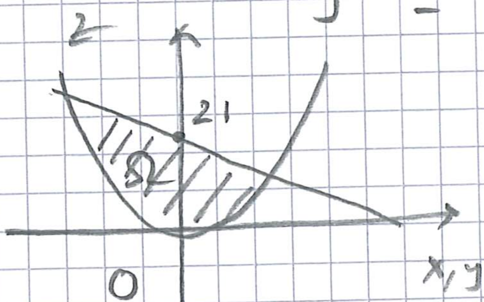
$$f'_y = 4y + c(x+y^2) + c'(x+y^2)\cdot 2y^2$$

$$f'_y(x,0) = 0 + c(x) + 0 = e^x \Rightarrow$$

$$\underline{f(x,y) = 2y^2 + y\cdot e^{x+y^2}}$$

7 Finn volymen av kroppen som avgränsas av

$z = 21-4y$ & $z = x^2+y^2$.



$$V = \iiint_{\Omega} 1 \, dx\,dy\,dz =$$

$$= \iint_{D} \left((21-4y) - (x^2+y^2)\right) dx\,dy;$$

$$D: \begin{cases} z = 21-4y \\ z = x^2+y^2 \end{cases} \Rightarrow \begin{aligned} &21-4y = x^2+y^2 \Leftrightarrow \\ &x^2+y^2+4y-21 = 0 \Leftrightarrow \end{aligned}$$

$$x^2+(y+2)^2-4-21 = 0 \Leftrightarrow \underline{x^2+(y+2)^2 = 5^2}$$

$$\Rightarrow \quad D: \quad \underline{x^2+(y+2)^2 \leq 5^2.}$$

$$\Rightarrow V = \iint\limits_{x^2+(y+2)^2 \le 5^2} \left(25 - x^2 - (y+2)^2\right) dx\,dy = \left| \begin{array}{l} u = x \\ v = y+2 \\ \frac{d\,uv}{d\,xy} = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \end{array} \right|$$

$$\Rightarrow V = \iint\limits_{u^2+v^2 \le 5^2} \left(25 - u^2 - v^2\right) du\,dv = \left| \begin{array}{l} u = r\cos\varphi \\ v = r\sin\varphi \end{array} \right|$$

$$= \int_0^{2\pi} \left(\int_0^5 \left(25 - r^2\right) \cdot r\,dr \right) d\varphi =$$

$$2\pi \int_0^5 \left(25r - r^3\right) dr = 2\pi \cdot \left(\frac{25 \cdot r^2}{2} - \frac{r^4}{4} \right) \Big|_0^5 =$$

$$= 2\pi \cdot \left(\frac{25 \cdot 25}{2} - \frac{(25)^2}{4} \right) = \frac{2\pi \cdot (25)^2}{4} = \underline{\frac{625\pi}{2}}$$