Matematiska institutionen, Linköpings universitet / Göran Bergqvist

TATM38, Exercises on time continuous models

1D population models

Consider the population models $\frac{dN}{dt} = Ng(N)$ for the following expressions of $g(N)$. Find the steady states. Determine their local stability. Draw the phase line. Sketch the solution $N(t)$ as a function of t for different initial values $N(0)$. What happens as $t \to \infty$?

Note: by separation of variables, you might be able to find an explicit solution, but it is not needed in order to answer the above questions.

1. $g(N) = \frac{r}{\alpha + N}$, $r > 0$ and $\alpha > 0$ constants

2. $g(N) = -\alpha \ln N$, $\alpha > 0$ constant (Gompertz model for tumor growth)

3. $g(N) = r(K - N)(N - M)$, $r > 0$ and $0 < M < K$ constants (model with allee effect).
For what value of $N(t)$ is the relative growth $N'(t)/N(t)$ maximal?

Steady states, Jacobian, linear systems

Numbers in parenthesis refer to the book E-K.

4. (4.16c)

Find all steady states of $\begin{cases} x' = x - x^2 - xy = F(x,y) \\ y' = y(1-y) = G(x,y) \end{cases}$ and determine the Jacobians at these points.

5. (4.18a with extras)

a) Solve $x''(t) + 3x'(t) + 2x(t) = 0$ with initial conditions $x(0) = 1$, $x'(0) = 1$

b) Put $y(t) = x'(t)$ and write the equation as a system $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$

What does $\mathrm{Tr}A$ and $\det A$ say about solutions as $t \to \infty$?

Solve the system by finding eigenvalues and eigenvectors of A. Impose also the initial conditions.

6. (4.22d with extras)

a) Solve the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

b) Find a second-order ODE for $x(t)$ from the system and solve it.

7. Solve the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Write the solution on real form.

Phase planes of linear systems

The problems 8 - 12 are from 5.7 in the book E-K or similar. For each of the linear systems below,

a) what does the trace and determinant of the matrix imply?

b) classify the stability characteristic of the steady state at (0,0)

c) for large t, what can you say about the solution?

d) find the solution curve that begins at $(x(0), y(0)) = (1, 0)$

e) plot the phase plane with Wolfram Alpha or some other program. In Wolfram Alpha, for problem 8, use "streamplot (3x+2y,4x+y)" or, if you want to choose interval, e.g., "streamplot (3x+2y,4x+y) x=-3..3 y=-2..2"

8. $\begin{cases} x' = 3x + 2y \\ y' = 4x + y \end{cases} \quad \Leftrightarrow \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

9. $\begin{cases} x' = 2x + y \\ y' = x + 2y \end{cases}$

10. $\begin{cases} x' = -x + 4y \\ y' = -x - y \end{cases}$

11. $\begin{cases} x' = -2y \\ y' = x \end{cases}$

12.* $\begin{cases} x' = -3x + 4y \\ y' = -x + y \end{cases}$ (only one eigenvector, see notes on linear systems)

Phase planes for non-linear systems

(Problems 5.5-5.6 adbe in the book E-K.)

For the following dynamical systems; find all steady states and determine their stability. Draw a phase plane picture with nullclines and directions of the vector field.

13. $\begin{cases} x' = y^2 - x^2 \\ y' = x - 1 \end{cases}$

14. $\begin{cases} x' = -xy \\ y' = (1 + x)(1 - y) \end{cases}$

15. $\begin{cases} x' = x(y^2 - y) \\ y' = x - y \end{cases}$

16. $\begin{cases} x' = x^2 - y \\ y' = y^2 - x \end{cases}$

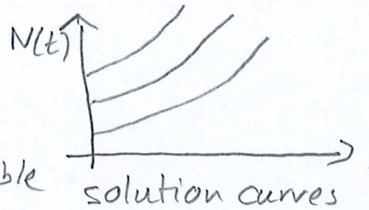
SOLUTIONS, 1D POPULATION MODELS

1. $N' = f(N) = Ng(N) = \frac{rN}{\alpha + N}$

Steady states: $\frac{r\bar{N}}{\alpha+\bar{N}} = 0 \Rightarrow \bar{N} = 0$. For $N > 0$ we have $N' = f(N) > 0$

Phase line

$N(t) \to \infty$, $t \to \infty$ (if $N(0) > 0$)
$\bar{N}=0$ unstable



solution curves

For N large: $N' = \frac{rN}{\alpha+N} \approx \frac{rN}{N} = r$ constant slope

Stability check with sign of $f'(\bar{N})$:

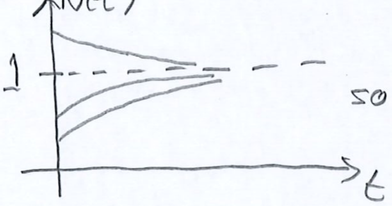
$$f'(N) = \frac{r(\alpha+N) - 1 \cdot rN}{(\alpha+N)^2} = \frac{r\alpha}{(\alpha+N)^2} \Rightarrow f'(0) = \frac{r\alpha}{\alpha^2} = \frac{r}{\alpha} > 0 \Rightarrow \bar{N} = 0 \text{ unstable}$$

2. $N' = f(N) = Ng(N) = -\alpha N \ln N$ ($N > 0$ but $N \ln N \to 0$, $N \to 0^+$)

Steady states $\bar{N} \ln \bar{N} = 0 \Rightarrow \bar{N} = 1$

Phase line

$N(t) \to 1$, $t \to \infty$, $\bar{N} = 1$ stable

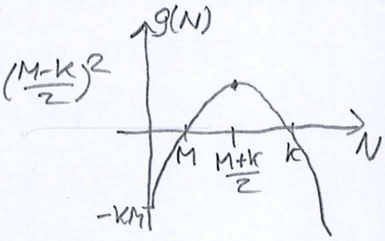


solution curves

Extra check: $f'(N) = -\alpha \ln N - \alpha N \cdot \frac{1}{N} = -\alpha \ln N - \alpha \Rightarrow f'(1) = -\alpha < 0 \Rightarrow \bar{N} = 1$ stable

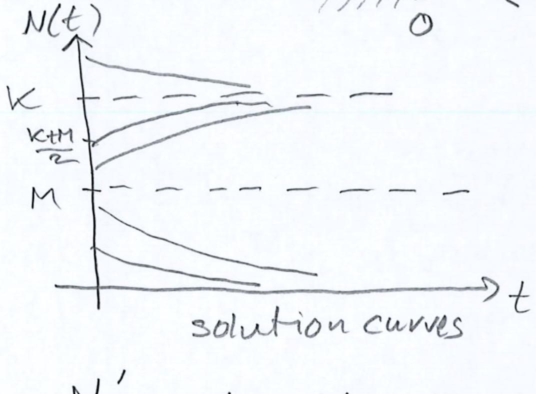
3. $N' = f(N) = Ng(N) = rN(K-N)(N-M)$, $r > 0$, $0 < M < K$

$$g(N) = (K-N)(N-M) = -N^2 - (M+K)N - KM = -\left(N - \frac{M+K}{2}\right)^2 + \left(\frac{M-K}{2}\right)^2$$



Steady states: $r\bar{N}(K-\bar{N})(\bar{N}-M) = 0 \Rightarrow \bar{N}_1 = 0, \bar{N}_2 = M, \bar{N}_3 = K$

Phase line



solution curves

$$N(t) \to \begin{cases} K & \text{if } N(0) > M \\ 0 & \text{if } N(0) < M \\ M & \text{if } N(0) = M \end{cases}$$

$\bar{N}_1 = 0$ and $\bar{N}_3 = K$ stable
$\bar{N}_2 = M$ unstable

$$\frac{N'}{N} = r(K-N)(N-M) = g(N) \text{ is maximal for } N = \frac{M+K}{2}$$

Extra check $f'(N) = r(K-N)(N-M) - rN(N-M) + rN(K-N) \Rightarrow$

$f'(0) = -rKM < 0 \Rightarrow \bar{N}_1 = 0$ stable
$f'(M) = rM(K-M) > 0 \Rightarrow \bar{N}_2 = M$ unstable
$f'(K) = -rK(K-M) < 0 \Rightarrow \bar{N}_3 = K$ stable

Solutions, steady states, Jacobian, linear systems

4. Steady states are given by $\begin{cases} F(x,y) = x - x^2 - xy = 0 & (1) \\ G(x,y) = y(1-y) = 0 & (2) \end{cases}$

(2) $\Rightarrow y = 0$ or $y = 1$. $y = 0$ in (1) $\Rightarrow x = 0$ or $x = 1$. $y = 1$ in (1) $\Rightarrow x = 0$. We get 3 steady states, $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (1,0)$, $(x_3, y_3) = (0,1)$.

The Jacobian matrix is $J(x,y) = \begin{pmatrix} F'_x & F'_y \\ G'_x & G'_y \end{pmatrix} = \begin{pmatrix} 1-2x-y & -x \\ 0 & 1-2y \end{pmatrix}$

At the steady states, $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$, $J(0,1) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

5. a) Characteristic equation is $r^2 + 3r + 2 = 0 \Rightarrow r_1 = -1,\ r_2 = -2 \Rightarrow$
the general solution is $x(t) = c_1 e^{-t} + c_2 e^{-2t}$.

The initial conditions [use $x'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$] give $\begin{cases} x(0) = c_1 + c_2 = 1 \\ x'(0) = -c_1 - 2c_2 = 1 \end{cases} \Rightarrow c_1 = 3,\ c_2 = -2$

and the solution is $x(t) = 3e^{-t} - 2e^{-2t}$.

b) $y = x' \Rightarrow x'' = y'$ and $x'' = -3x' - 2x \Leftrightarrow y' = -3y - 2x$.

The ODE for $x(t)$ is equivalent to the system $\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}_{=A} \begin{pmatrix} x \\ y \end{pmatrix}$

$\mathrm{tr}A = -3 < 0$ and $\det A = 2 > 0$ imply that solutions $\to 0$ as $t \to \infty$.

Eigenvalues to A: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = -2$

Eigenvectors : $(A + I)\bar{v}_1 = \bar{0} \Rightarrow \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \bar{v}_1 = \bar{0} \ \Rightarrow\ \bar{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(A + 2I)\bar{v}_2 = \bar{0} \Rightarrow \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \bar{v}_2 = \bar{0} \ \Rightarrow\ \bar{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

General solution to the system is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

The initial conditions $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} k_1 + k_2 \\ -k_1 - 2k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow$

$k_1 = 3,\ k_2 = -2$ and the solution is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 3e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - 2e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

On the first row here we see $x(t) = 3e^{-t} - 2e^{-2t}$ as before.

6. a) Eigenvalues : $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 4 \\ -2 & 5-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = 1$

Eigenvectors : $(A - 3I)\bar{v}_1 = \bar{0} \Rightarrow \begin{pmatrix} -4 & 4 \\ -2 & 2 \end{pmatrix} \bar{v}_1 = \bar{0} \ \Rightarrow\ \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A - I)\bar{v}_2 = \bar{0} \Rightarrow \begin{pmatrix} -2 & 4 \\ -2 & 4 \end{pmatrix} \bar{v}_2 = \bar{0} \ \Rightarrow\ \bar{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

General solution to the system is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ or $\begin{cases} x(t) = c_1 e^{3t} + 2c_2 e^{t} \\ y(t) = c_1 e^{3t} + c_2 e^{t} \end{cases}$

b) $x' = -x + 4y \Rightarrow x'' = -x' + 4y' = -x' + 4(-2x + 5y) = -x' - 8x + 20 \cdot \dfrac{x' + x}{4} = 4x' - 3x \Rightarrow$
the ODE for $x(t)$ is $x''(t) - 4x'(t) + 3x(t) = 0$.
Characteristic equation is $r^2 - 4r + 3 = 0 \Rightarrow r_1 = 3,\ r_2 = 1 \Rightarrow$
the general solution is $x(t) = k_1 e^{3t} + k_2 e^{t}$ (note that compared to above $k_1 = c_1$ but $k_2 = 2c_2$).

7. Eigenvalues : $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 5 \\ -2 & 5-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = 2 \pm i$

Eigenvectors : $(A - (2+i)I)\bar{v}_1 = \bar{0} \Rightarrow \begin{pmatrix} -3-i & 5 \\ -2 & 3-i \end{pmatrix} \bar{v}_1 = \bar{0} \ \Rightarrow\ \bar{v}_1 = \begin{pmatrix} 5 \\ 3+i \end{pmatrix}$

$(A - (2-i))\bar{v}_2 = \bar{0} \Rightarrow \begin{pmatrix} -3+i & 5 \\ -2 & 3+i \end{pmatrix} \bar{v}_2 = \bar{0} \ \Rightarrow\ \bar{v}_2 = \begin{pmatrix} 5 \\ 3-i \end{pmatrix}$

General solution to the system is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{(2+i)t}\begin{pmatrix} 5 \\ 3+i \end{pmatrix} + c_2 e^{(2-i)t}\begin{pmatrix} 5 \\ 3-i \end{pmatrix} =$

$= e^{2t}[c_1(\cos t + i\sin t)(\begin{pmatrix} 5 \\ 3 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}) + c_2(\cos t - i\sin t)(\begin{pmatrix} 5 \\ 3 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix})] =$

$= e^{2t}[\underbrace{(c_1+c_2)}_{=k_1}(\cos t\begin{pmatrix} 5 \\ 3 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}) + \underbrace{i(c_1-c_2)}_{=k_2}(\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin t\begin{pmatrix} 5 \\ 3 \end{pmatrix})] =$

$= e^{2t}\begin{pmatrix} 5k_1\cos t + 5k_2\sin t \\ (3k_1+k_2)\cos t + (3k_2-k_1)\sin t \end{pmatrix}$ (real expression if k_1 and k_2 real)

Hence, we get $\begin{cases} x(t) = 5c_1e^{(2+i)t} + 5c_2e^{(2-i)t} = e^{2t}(5k_1\cos t + 5k_2\sin t) \\ y(t) = (3+i)c_1e^{(2+i)t} + (3-i)c_2e^{(2-i)t} = e^{2t}((3k_1+k_2)\cos t + (3k_2-k_1)\sin t) \end{cases}$

Solutions, phase planes of linear systems (plots to 8-12 follow after the solution of 12)

8. $\det A = -5 < 0 \Rightarrow \lambda_1 > 0, \lambda_2 < 0 \Rightarrow$ saddle point

Eigenvalues to A: $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 = 0 \Rightarrow \lambda_1 = 5 > 0, \lambda_2 = -1 < 0 \Rightarrow$ saddle point

Eigenvectors : $(A-5I)\bar{v}_1 = \bar{0} \Rightarrow \begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix}\bar{v}_1 = \bar{0} \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A+I)\bar{v}_2 = \bar{0} \Rightarrow \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix}\bar{v}_2 = \bar{0} \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

General solution to the system is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ or $\begin{cases} x(t) = c_1e^{5t} + c_2e^{-t} \\ y(t) = c_1e^{5t} - 2c_2e^{-t} \end{cases}$

For large t, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx c_1e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (if $c_1 \neq 0$), so $y \approx x$

The initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} c_1+c_2 \\ c_1-2c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \Rightarrow \begin{cases} x(t) = \frac{2}{3}e^{5t} + \frac{1}{3}e^{-t} \\ y(t) = \frac{2}{3}e^{5t} - \frac{2}{3}e^{-t} \end{cases}$

9. $\mathrm{Tr}A = 4 > 0$, $\det A = 3 > 0 \Rightarrow \mathrm{Re}\lambda_{1,2} > 0 \Rightarrow$ unstable (not yet clear if spiral)

Eigenvalues to A: $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda_1 = 3 > 0, \lambda_2 = 1 > 0 \Rightarrow$ unstable and no spiral

Eigenvectors : $(A-3I)\bar{v}_1 = \bar{0} \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\bar{v}_1 = \bar{0} \Rightarrow \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(A-I)\bar{v}_2 = \bar{0} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\bar{v}_2 = \bar{0} \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

General solution to the system is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $\begin{cases} x(t) = c_1e^{3t} + c_2e^{t} \\ y(t) = c_1e^{3t} - c_2e^{t} \end{cases}$

For large t, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is dominated by $c_1e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (if $c_1 \neq 0$), so $y/x \approx 1$

The initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} c_1+c_2 \\ c_1-c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \Rightarrow \begin{cases} x(t) = \frac{1}{2}(e^{3t} + e^{t}) \\ y(t) = \frac{1}{2}(e^{3t} - e^{t}) \end{cases}$

10. $\mathrm{Tr}A = -2 < 0$, $\det A = 5 > 0 \Rightarrow \mathrm{Re}\lambda_{1,2} < 0 \Rightarrow$ stable (not yet clear if spiral)

Eigenvalues to A: $\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 4 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = -1 \pm 2i \Rightarrow$ stable spiral ($\mathrm{Re}\lambda_{1,2} < 0$)

Eigenvectors : $(A-(-1+2i)I)\bar{v}_1 = \bar{0} \Rightarrow \begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix}\bar{v}_1 = \bar{0} \Rightarrow \bar{v}_1 = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$

$(A-(-1-2i)I)\bar{v}_2 = \bar{0} \Rightarrow \begin{pmatrix} 2i & 4 \\ -1 & 2i \end{pmatrix}\bar{v}_2 = \bar{0} \Rightarrow \bar{v}_2 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$

General solution to the system is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{(-1+2i)t} \begin{pmatrix} -2i \\ 1 \end{pmatrix} + c_2 e^{(-1-2i)t} \begin{pmatrix} 2i \\ -1 \end{pmatrix} =$

$= e^{-t} \begin{pmatrix} 2k_1 \sin 2t - 2k_2 \cos 2t \\ k_1 \cos 2t + k_2 \sin 2t \end{pmatrix}$ or $\begin{cases} x(t) = e^{-t}(2k_1 \sin 2t - 2k_2 \cos 2t) \\ y(t) = e^{-t}(k_1 \cos 2t + k_2 \sin 2t) \end{cases}$ $(k_1 = c_1 + c_2, k_2 = i(c_1 - c_2))$

For large t, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ along a spiral

The initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} -2k_2 \\ k_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \Rightarrow \begin{cases} x(t) = e^{-t} \cos 2t \\ y(t) = -\frac{1}{2} e^{-t} \sin 2t \end{cases}$

11. $\text{Tr}A = 0$, $\det A = 2 > 0 \Rightarrow$ neutral steady state

Eigenvalues to A: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 2 = 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{2} \Rightarrow$ neutral

Eigenvectors : $(A - (i\sqrt{2})I)\bar{v}_1 = \bar{0} \Rightarrow \begin{pmatrix} -i\sqrt{2} & -2 \\ 1 & -i\sqrt{2} \end{pmatrix} \bar{v}_1 = \bar{0} \Rightarrow \bar{v}_1 = \begin{pmatrix} i\sqrt{2} \\ 1 \end{pmatrix}$

$(A - (-i\sqrt{2})I)\bar{v}_2 = \bar{0} \Rightarrow \begin{pmatrix} i\sqrt{2} & -2 \\ 1 & i\sqrt{2} \end{pmatrix} \bar{v}_2 = \bar{0} \Rightarrow \bar{v}_2 = \begin{pmatrix} -i\sqrt{2} \\ 1 \end{pmatrix}$

General solution to the system is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{i\sqrt{2}t} \begin{pmatrix} i\sqrt{2} \\ 1 \end{pmatrix} + c_2 e^{-i\sqrt{2}t} \begin{pmatrix} -i\sqrt{2} \\ 1 \end{pmatrix} =$

$= \begin{pmatrix} -\sqrt{2}k_1 \sin \sqrt{2}t + \sqrt{2}k_2 \cos \sqrt{2}t \\ k_1 \cos \sqrt{2}t + k_2 \sin \sqrt{2}t \end{pmatrix}$ $(k_1 = c_1 + c_2,\ k_2 = i(c_1 - c_2))$

For all t, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ moves along the ellipse $x^2 + 2y^2 = 2(k_1^2 + k_2^2)$ (calculate $x^2 + 2y^2$!), one loop in $\frac{2\pi}{\sqrt{2}}$ time units. To see that $x^2 + 2y^2$ is constant for solutions:

$\frac{d}{dt}(x^2 + 2y^2) = 2xx' + 4yy' = 2x(-2y) + 4yx = 0$

The initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \sqrt{2}k_2 \\ k_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \end{pmatrix} \Rightarrow \begin{cases} x(t) = \cos \sqrt{2}t \\ y(t) = \frac{1}{\sqrt{2}} \sin \sqrt{2}t \end{cases}$

12. $\text{Tr}A = -2 < 0$, $\det A = 1 > 0 \Rightarrow$ stable (not yet clear if spiral)

Eigenvalues to A: $\det(A - \lambda I) = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1 < 0 \Rightarrow$ stable and no spiral

Eigenvectors : $(A + I)\bar{v}_1 = \bar{0} \Rightarrow \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \bar{v}_1 = \bar{0} \Rightarrow \bar{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (only one eigenvector)

Take, e.g., $\bar{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow (A + I)\bar{v}_2 = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = (-1)\bar{v}_1 \Rightarrow$ (see notes)

General solution to the system is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t}[(c_2 - c_1 t) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}] = e^{-t} \begin{pmatrix} c_1 + 2c_2 - 2c_1 t \\ c_2 - c_1 t \end{pmatrix}$

For large t, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \approx -c_1 t e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

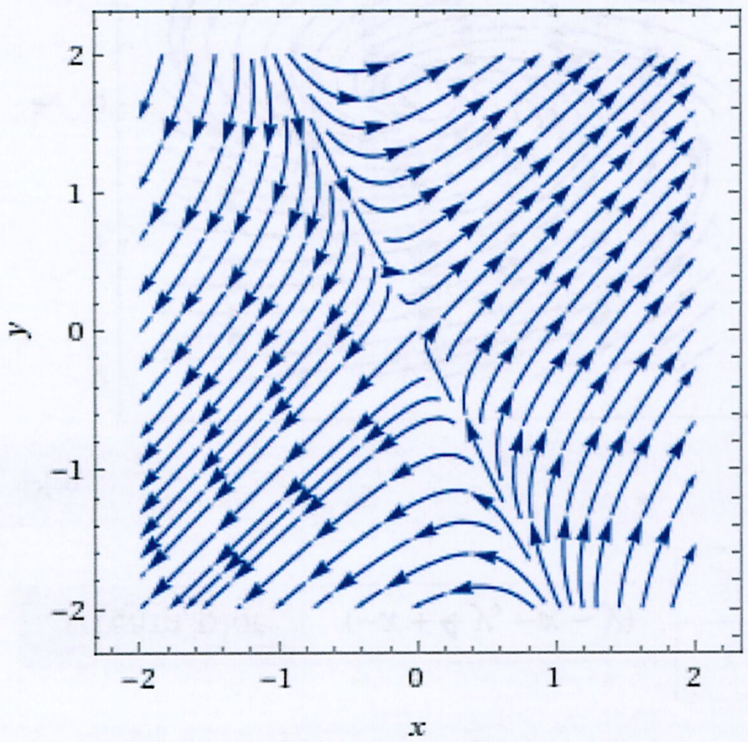
The initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} c_1 + 2c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x(t) = (1 - 2t)e^{-t} \\ y(t) = -te^{-t} \end{cases}$

Plot 8.

Input interpretation:

stream plot	$(3x+2y, 4x+y)$	$x = -2$ to 2 $y = -2$ to 2

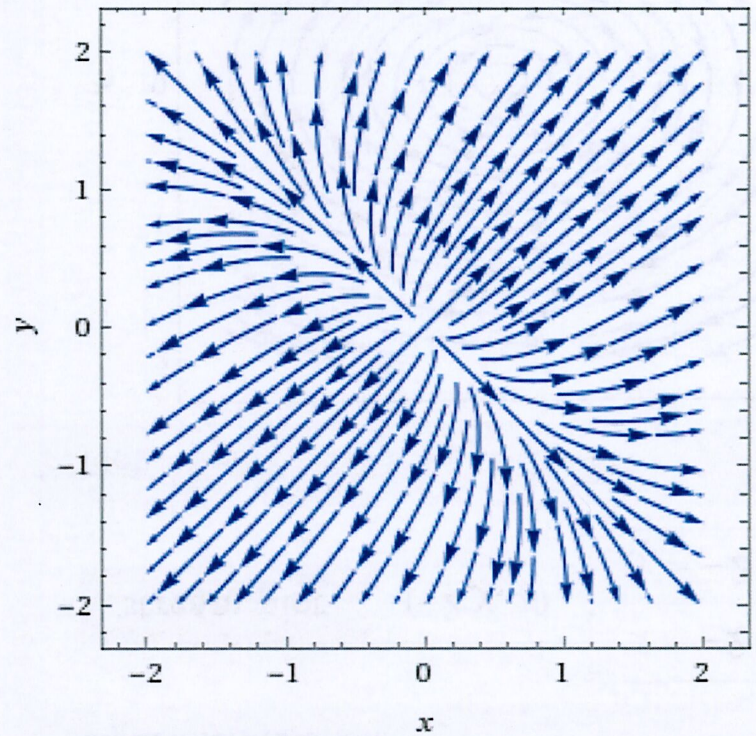
Plot:



9.

stream plot	$(2x+y, x+2y)$	$x = -2$ to 2 $y = -2$ to 2

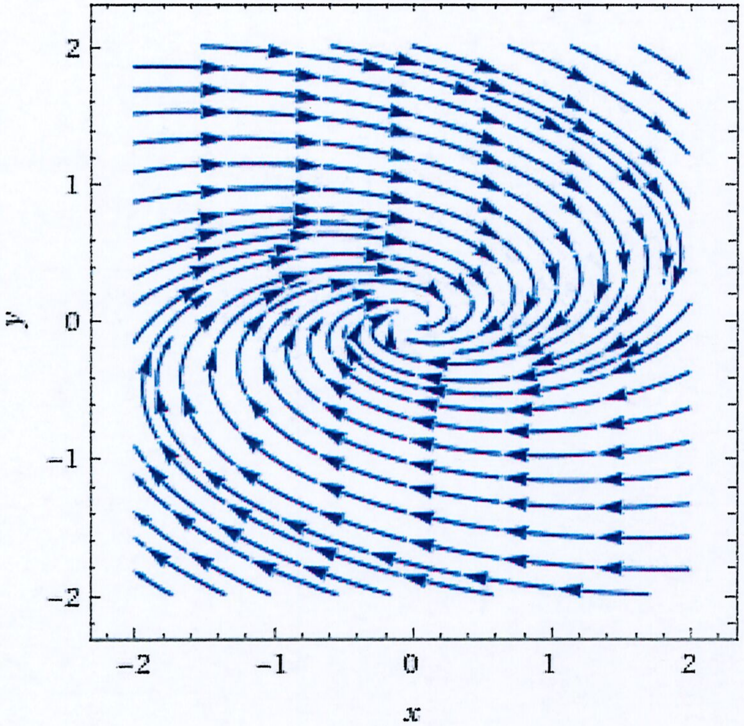
Plot:



10.

stream plot	$(-x+4y, -x-y)$	$x = -2$ to 2
		$y = -2$ to 2

Plot:

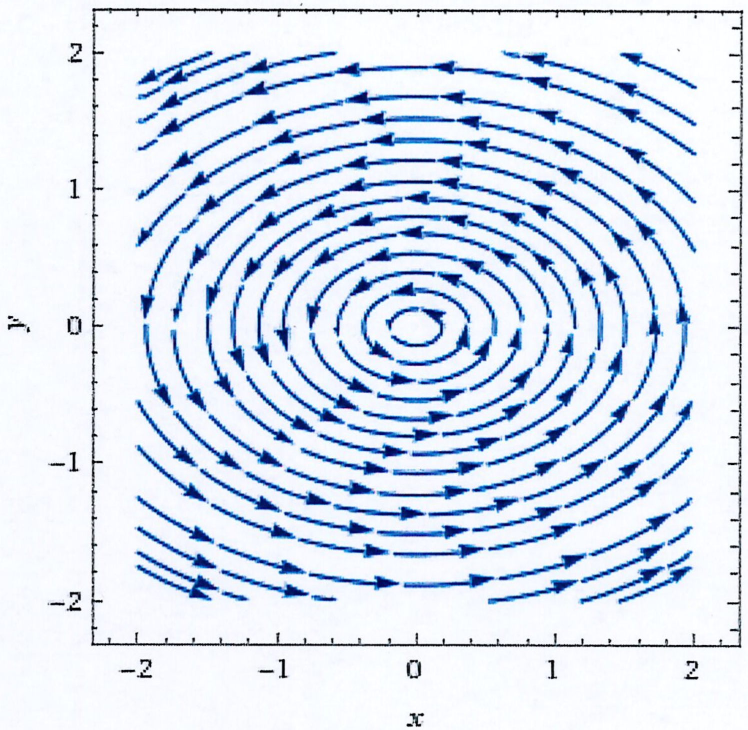


11.

Input interpretation:

stream plot	$(-2y, x)$	$x = -2$ to 2
		$y = -2$ to 2

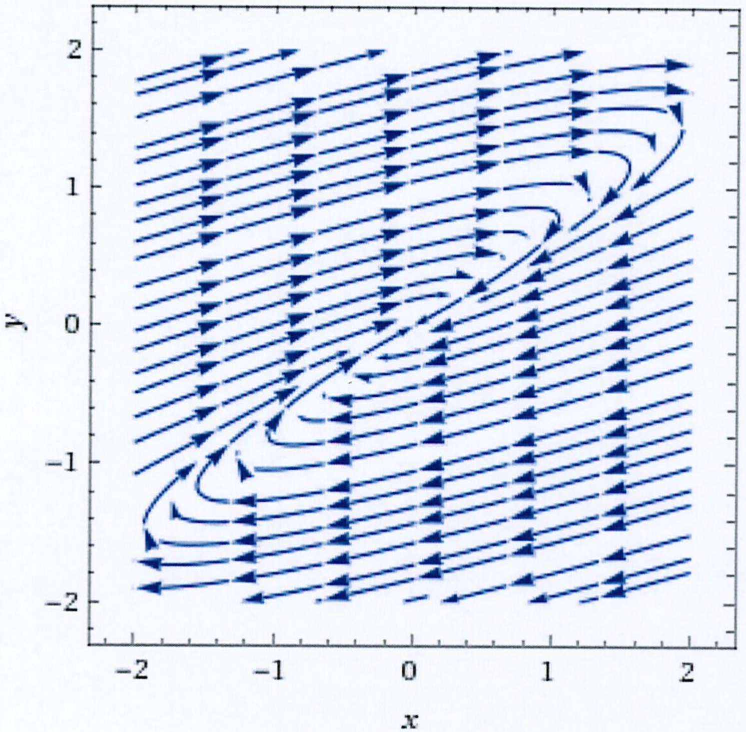
Plot:



12.

stream plot	$(-3x+4y, -x+y)$	$x = -2$ to 2 $y = -2$ to 2

Plot:

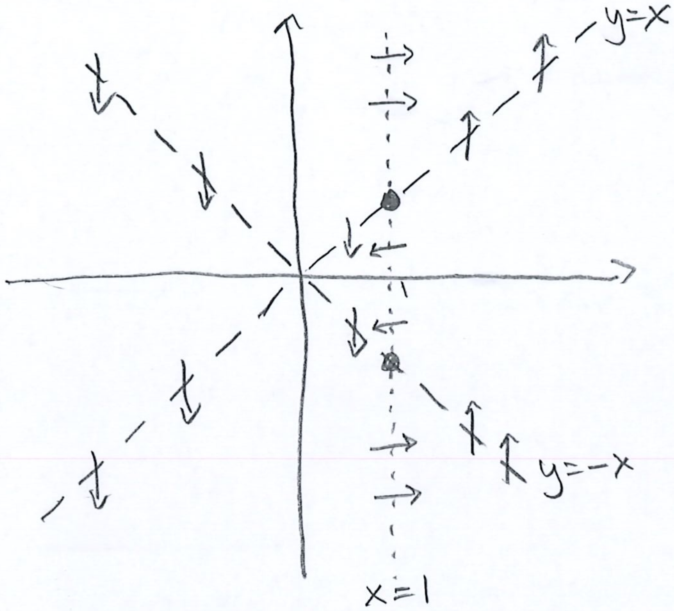


Solutions, phase planes for non-linear systems (separate plots after)

13. (EK 5.5-6a) $\begin{cases} x' = y^2 - x^2 = F_1(x,y) \\ y' = x - 1 = F_2(x,y) \end{cases}$

x nullclines $F_1(x,y) = y^2 - x^2 = 0 \Rightarrow y = \pm x$ two lines

y nullclines $F_2(x,y) = x - 1 = 0 \Rightarrow x = 1$ vertical line



Steady states $\begin{cases} F_1(x,y) = 0 \\ F_2(x,y) = 0 \end{cases}$

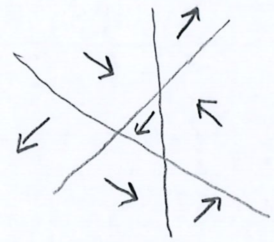
$\Rightarrow (x_1, y_1) = (1,1), (x_2, y_2) = (1,-1)$

Draw $\bar{F} = (F_1, F_2)$ on nullclines

On $F_1 = 0$, $\bar{F} = \underbrace{(0, F_2)}_{\text{vertical}}$, $F_2 \begin{cases} > 0 \text{ if } x > 1 \\ < 0 \text{ if } x < 1 \end{cases}$

On $F_2 = 0$, $\bar{F} = \underbrace{(F_1, 0)}_{\text{horizontal}}$, $F_1 \begin{cases} > 0 \text{ if } y^2 > x^2 \\ < 0 \text{ if } y^2 < x^2 \end{cases}$

(1,1) looks unstable (saddle)
(1,-1) unclear, some spiral



directions of $\bar{F}$

Check also wolfram plot

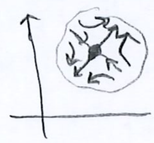
Check Jacobians

$$J(x,y) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -2x & 2y \\ 1 & 0 \end{pmatrix}$$

At $(x_1, y_1) = (1,1)$: $J(1,1) = \begin{pmatrix} -2 & 2 \\ 1 & 0 \end{pmatrix} = J_1$, $\det J_1 = -2 < 0 \Rightarrow$ saddle (unstable)

or $\lambda_{1,2} = -1 \pm \sqrt{3}$, $\lambda_1 > 0, \lambda_2 < 0 \Rightarrow$ saddle

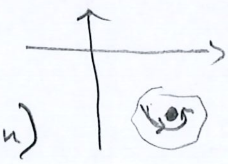
(can study eigenvectors to get more info.)



At $(x_2, y_2) = (1,-1)$: $J(1,-1) = \begin{pmatrix} -2 & -2 \\ 1 & 0 \end{pmatrix}$, eigenvalues $\lambda_{1,2} = -1 \pm i$

$\text{Re}(\lambda_{1,2}) = -1 < 0 \Rightarrow$ stable spiral

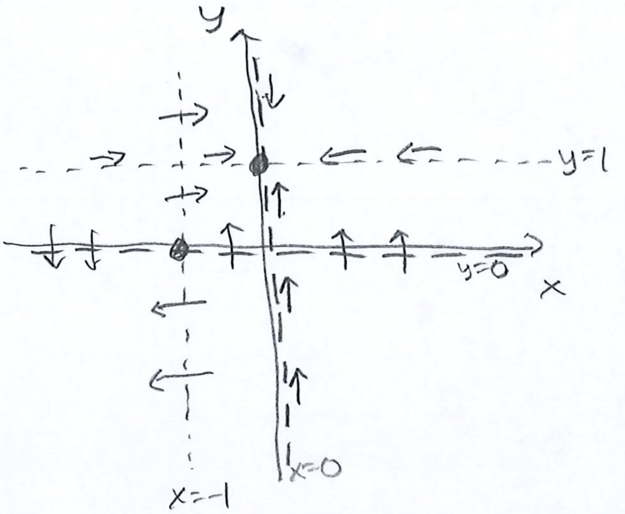
(difficult to see spiral on plots → try to zoom in)



14. (EK 5.5-6d) $\begin{cases} x' = -xy = F_1(x,y) \\ y' = (1+x)(1-y) = F_2(x,y) \end{cases}$

x nullclines $\quad -xy = 0 \Rightarrow x = 0$ or $y = 0$ (coordinates axes)

y nullclines $\quad (1+x)(1-y) = 0 \Rightarrow x = -1$ or $y = 1$ (two lines)



Two steady states: $(0,1)$, $(-1,0)$

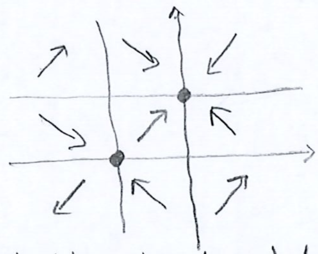
Draw

$\bar{F} = (0, F_2) = (0, (1+x)(1-y))$ on $x=0$ and $y=0$

$\bar{F} = (F_1, 0) = (-xy, 0)$ on $x=-1$ and $y=1$

$(0,1)$ looks stable

$(-1,0)$?



$$J(x,y) = \begin{pmatrix} -y & -x \\ 1-y & -1-x \end{pmatrix}$$

$J(0,1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = -1 < 0 \Rightarrow (0,1)$ stable steady state

(all vectors are eigenvectors)

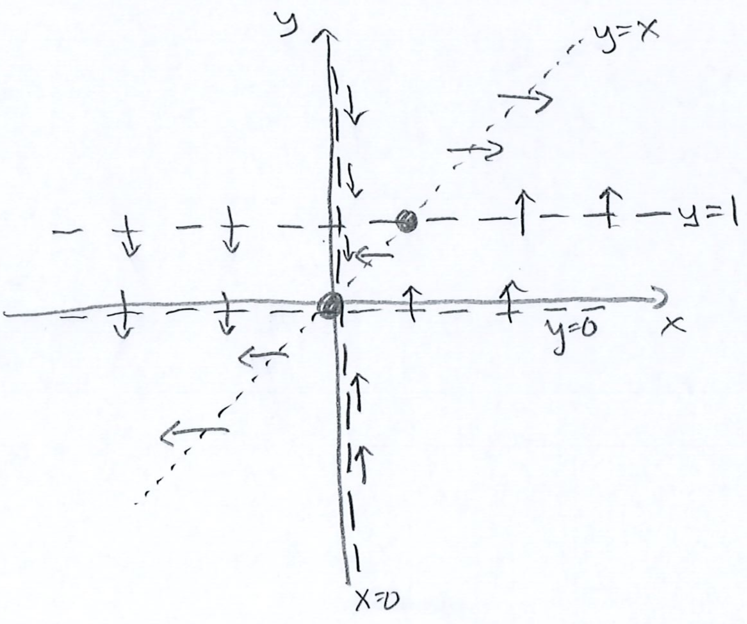
$J(-1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda_1 = 1 > 0,\ \lambda_2 = -1 < 0 \Rightarrow (-1,0)$ saddle (unstable)

see also Wolfram plot

15. (EK 5.5-6b) $\begin{cases} x' = x(y^2 - y) \\ y' = x - y \end{cases}$

x nullclines $\quad x(y^2-y) = 0 \Rightarrow x = 0, y = 0, y = 1$, 3 lines (2 are coord. axes)

y nullclines $\quad x - y = 0 \Rightarrow y = x$, line

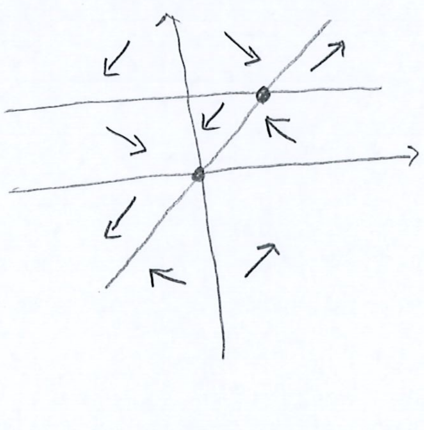


Steady states $(0,0)$, $(1,1)$

Draw

$\bar{F} = (0, x-y)$ on $y=0, y=1, x=0$

$\bar{F} = (xy(y-1), 0)$ on $y = x$



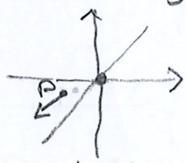
3 cont. $J(x,y) = \begin{pmatrix} y^2-y & x(2y-1) \\ 1 & -1 \end{pmatrix} \Rightarrow$

$J(1,1) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$, $\lambda_1 > 0, \lambda_2 < 0 \Rightarrow$ saddle (unstable)

$J(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \Rightarrow \underline{\lambda_1 = 0}$, $\lambda_2 = -1$. $\lambda_1 = 0 \Rightarrow$ unclear, could be stable or like a saddle

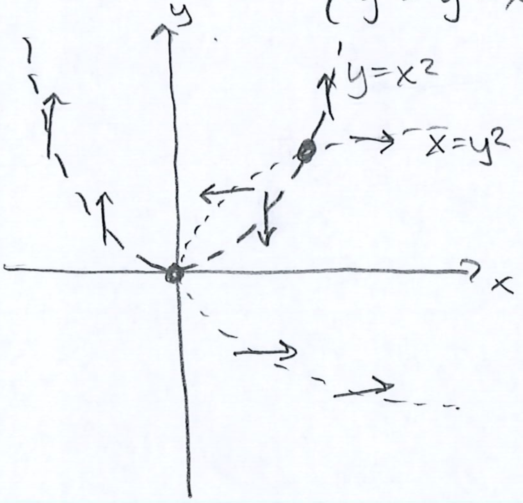
Picture indicates that if $x(0)$ and $y(0)$ negative, and $x(0)$ more negative, we move away from $(0,0)$. Take $P = (-2\varepsilon, -\varepsilon)$

At P: $(x', y') = (\underbrace{-2\varepsilon(\varepsilon^2+\varepsilon)}_{<0}, \underbrace{-2\varepsilon+\varepsilon}_{<0})$ for all $\varepsilon > 0$ $\Rightarrow$ we move away from $(0,0)$



$\Rightarrow (0,0)$ unstable

16. (EK 5.5-6 e) $\begin{cases} x' = x^2 - y \\ y' = y^2 - x \end{cases}$ x nullcline $y = x^2$, y nullcline $x = y^2$

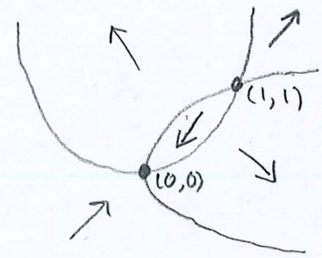


Steady states at $(0,0)$, $(1,1)$

Draw

$\bar{F} = (0, y^2 - x)$ on $y = x^2$

$\bar{F} = (x^2 - y, 0)$ on $x = y^2$



$J(x,y) = \begin{pmatrix} 2x & -1 \\ -1 & 2y \end{pmatrix}$

$J(1,1) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = J_1$, $Tr\, J_1 = 4 > 0$, $\det J_1 = 3 > 0 \Rightarrow$ unstable (or $\lambda_1 = 3 > 0$, $\lambda_2 = 1 > 0 \Rightarrow$ unstable)

$J(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = J_2$, $\det J_2 = -1 < 0 \Rightarrow$ saddle (unstable)

$\lambda_1 = 1$, $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ outgoing, $\lambda_2 = -1$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ incoming

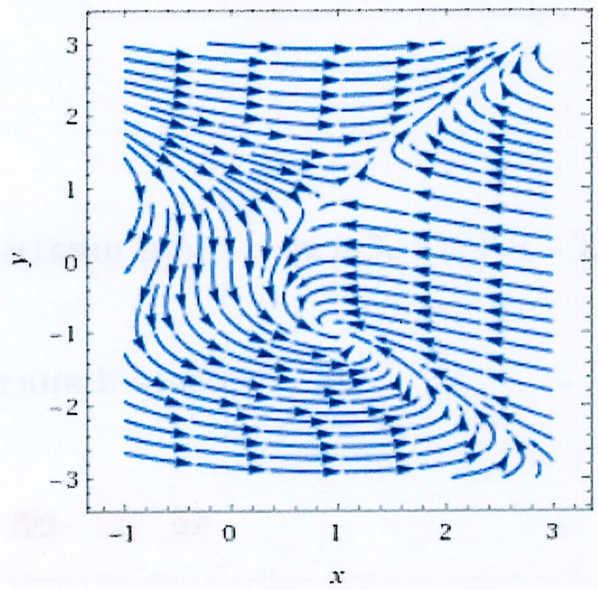
Plot 13.

streamplot[{y^2-x^2,x-1},{x,-1,3},{y,-3,3}]

Input interpretation:

stream plot	$(-x^2 + y^2, -1 + x)$	$x = -1$ to 3 $y = -3$ to 3

Plot:



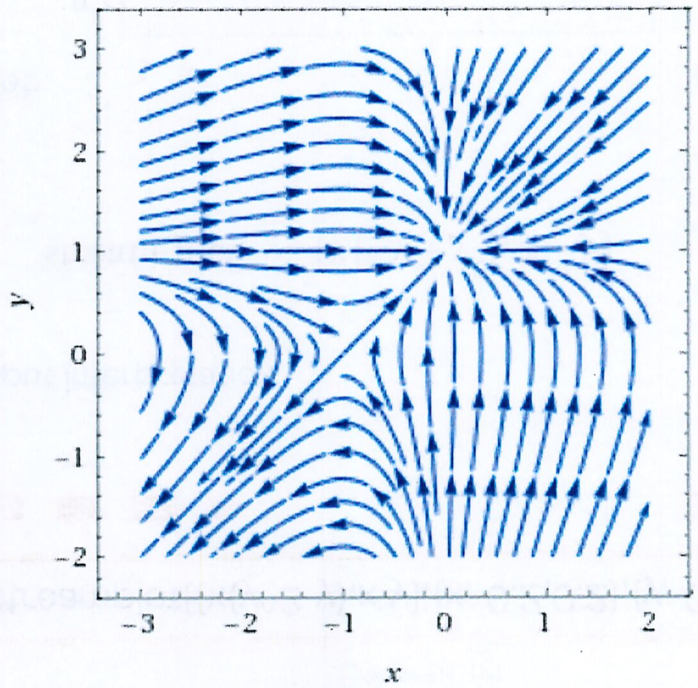
14.

streamplot[{-xy,(1+x)(1-y)},{x,-3,2},{y,-2,3}]

Input interpretation:

stream plot	$(-x\,y, (1 + x)(1 - y))$	$x = -3$ to 2 $y = -2$ to 3

Plot:



15.

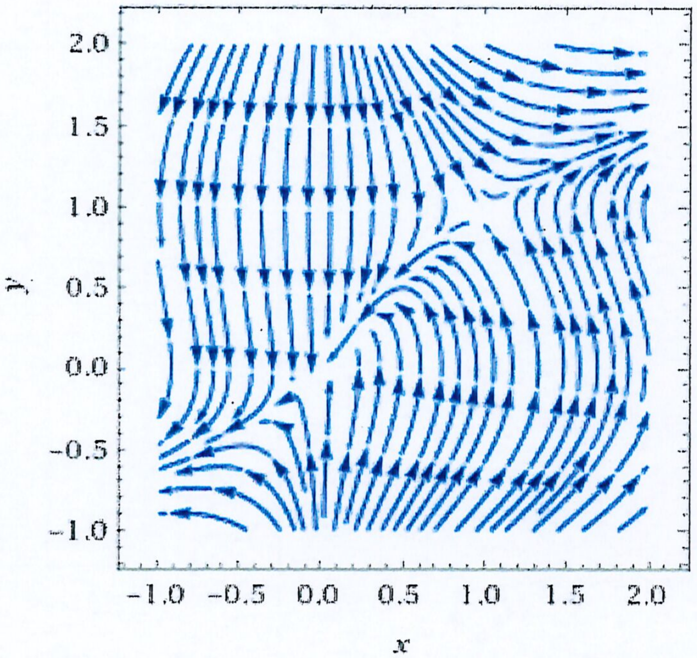
streamplot[{x(y^2-y),x-y},{x,-1,2},{y,-1,2}]

≡ Browse E

Input interpretation:

stream plot	$(x(-y+y^2), x-y)$	x = −1 to 2 y = −1 to 2

Plot:



15 zoomed in.

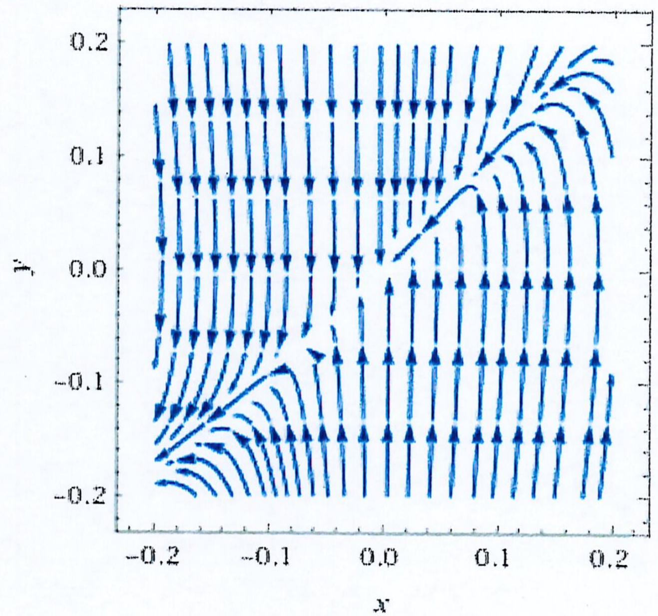
streamplot[{x(y^2-y),x-y},{x,-0.2,0.2},{y,-0.2,0.2}]

≡ Browse Examp

Input interpretation:

stream plot	$(x(-y+y^2), x-y)$	x = −0.2 to 0.2 y = −0.2 to 0.2

Plot:



16.

stream plot	$(x^2 - y, -x + y^2)$	$x = -2$ to 3 $y = -2$ to 3

Plot: